Continuous time optimal control

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Outline of today’s talk

- Continuous time optimisation problems.
- Euler-Lagrange Equations.
- Necessary conditions for general continuous time optimal control.
- The standard infinite horizon case, and the current and present value Hamiltonians.
- Assorted examples.
- Hamilton-Jacobi-Bellman (HJB) equations.
- Stochastic HJB equations.

Reading for today

- Relevant sections/appendices of any growth textbook. E.g.:
  - “Economic Growth”: Barro and Sala-i-Martin (Appendix A.1, A.3)
  - “Introduction to Modern Economic Growth”: Acemoglu (Chapter 7 and Appendix B)
  - “The Economics of Growth”: Aghion and Howitt
- “The Economics of Inaction: Stochastic Control Models with Fixed Costs”: Stokey (2008)
  - For the stochastic HJB material.
- “Solving the New Keynesian Model in Continuous Time”: Fernandez-Villaverde, Posch and Rubio-Ramirez (2012)
  - Great, readable example of what is possible in continuous time.

Example continuous time optimisation problem

- In continuous time macro, we are often interested in finding the paths for control variables c and state variables x that maximise objectives of the form:

$$\int_0^\infty e^{-\rho t} u\big(t, x(t), c(t)\big)\, dt$$

- given the initial value of the state, $x(0) = x_0$, where the state evolves according to:

$$\dot{x}(t) = f\big(t, x(t), c(t)\big),$$

- and where ρ is the discount rate.
  - Note: throughout this lecture dots above variables will denote derivatives with respect to time.

- For example, in a simple asset eating model, we might have, $u(t, x, c) = \log c$, and $f(t, x, c) = rx - c$, where r is the real interest rate.

The Euler-Lagrange Equation

- Suppose we seek to find the stationary points of the functional $S: (\mathbb{R} \to \mathbb{R}^n) \to \mathbb{R}$ given by:

$$S(z) = \int_0^T \mathcal{L}\big(t, z(t), \dot{z}(t)\big)\, dt\,.$$

  - where $\mathcal{L}$ has continuous first partial derivatives.
- Note:
  - A functional is a scalar valued function of a function, in this case the function in question $z: \mathbb{R} \to \mathbb{R}^n$.
  - Any maximum or minimum of S must also be a stationary point of it.
- The Euler-Lagrange equation states that if z is a stationary point of S, then for all $i \in 1, \dots, n$, and all $t \in (0, T)$:

$$\frac{\partial \mathcal{L}\big(t, z(t), \dot{z}(t)\big)}{\partial z_i(t)} = \frac{d}{dt} \frac{\partial \mathcal{L}\big(t, z(t), \dot{z}(t)\big)}{\partial \dot{z}_i(t)}\,.$$

- Or in more compact notation: $\mathcal{L}_2\big(t, z(t), \dot{z}(t)\big) = \frac{d}{dt} \mathcal{L}_3\big(t, z(t), \dot{z}(t)\big)$,
  - where, for a column vector valued function $f(x, y)$, and e.g. $y \in \mathbb{R}^n$, $f_2(x, y)$ is the matrix $\left[\frac{\partial f(x,y)}{\partial y_1} \quad \cdots \quad \frac{\partial f(x,y)}{\partial y_n}\right]$.

Transversality constraints

- Unless the terminal value of the state is specified in advance, then an additional condition is needed in order to pin down the path of the state and costate.
  - An example of a situation in which the terminal state is specified would be seeking the optimal rate at which to collect a given stock of assets.
- The necessary condition for optimality is that:

$$\mathcal{L}_3\big(T, z(T), \dot{z}(T)\big) = 0.$$

- Intuitively, this states that an extra unit of the state is worthless at the end of time.
  - This result is often combined with the Euler-Lagrange equation result under the banner of "Pontryagin's Maximum Principle".

- We view infinite horizon models as the limit of finite horizon ones as $T \to \infty$, thus in the infinite horizon case you might think that a necessary condition would be $\lim_{T\to\infty} \mathcal{L}_3\big(T, z(T), \dot{z}(T)\big) = 0$. In fact, this is not necessary.
- Kamihigashi (2001) shows that under weak assumptions the necessary transversality condition is:

$$\lim_{T\to\infty} \mathcal{L}_3\big(T, z(T), \dot{z}(T)\big) z(T) = 0$$

General continuous time optimal control (1/2)

- Suppose we seek to maximise:

$$\Gamma(x,c) = \Psi\big(T, x(T)\big) + \int_0^T \Upsilon\big(t, x(t), c(t), \dot{x}(t)\big)\, dt\,,$$

  - subject to: $0 = \Phi\big(t, x(t), c(t), \dot{x}(t)\big)$ and $x(0) = x_0$.
- First note that:

$$\frac{d}{dt}\Psi\big(t, x(t)\big) = \Psi_1\big(t, x(t)\big) + \Psi_2\big(t, x(t)\big)\dot{x}(t).$$

- So:

$$\int_0^T \big[\Psi_1\big(t, x(t)\big) + \Psi_2\big(t, x(t)\big)\dot{x}(t)\big]\, dt = \Psi\big(T, x(T)\big) - \Psi(0, x_0).$$

- Hence:

$$\Gamma(x,c) = \Psi(0, x_0) + \int_0^T \big[\Upsilon\big(t, x(t), c(t), \dot{x}(t)\big) + \Psi_1\big(t, x(t)\big) + \Psi_2\big(t, x(t)\big)\dot{x}(t)\big]\, dt\,.$$

- We proceed by applying the previous result to the Lagrangian:

$$\mathcal{L}(t, z, \dot{z}) = \Upsilon(t, x, c, \dot{x}) + \Psi_1(t, x) + \Psi_2(t, x)\dot{x} + \mu'\Phi(t, x, c, \dot{x}),$$

  - where $z = [x', c', \mu']'$.

General continuous time optimal control (2/2)

- From the previous result, the stationary points of $S(z)=\int_0^T \mathcal{L}\big(t,z(t),\dot{z}(t)\big)\,dt$ satisfy:

$$\Upsilon_2(t,x,c,\dot{x})+\Psi_{12}(t,x)+\Psi_{22}(t,x)\dot{x}+\mu'\Phi_2(t,x,c,\dot{x})$$
$$=\frac{d}{dt}\big(\Upsilon_4(t,x,c,\dot{x})+\Psi_2(t,x)+\mu'\Phi_4(t,x,c,\dot{x})\big),$$
$$\Upsilon_3(t,x,c,\dot{x})+\mu'\Phi_3(t,x,c,\dot{x})=0,$$
$$\Phi(t,x,c,\dot{x})=0,$$
$$\Upsilon_4\big(T,x(T),c(T),\dot{x}(T)\big)+\Psi_2\big(T,x(T)\big)+\mu'(T)\Phi_4\big(T,x(T),c(T),\dot{x}(T)\big)=0.$$

- Note that the penultimate equation implies that at a stationary point of S, the constraint is satisfied, as required.
- Also, note that the first equation simplifies to:

$$\Upsilon_2(t,x,c,\dot{x})+\mu'\Phi_2(t,x,c,\dot{x})=\frac{d}{dt}\big(\Upsilon_4(t,x,c,\dot{x})+\mu'\Phi_4(t,x,c,\dot{x})\big),$$

  - as $\Psi_{21}(t,x)=\Psi_{12}(t,x)$.

A standard infinite horizon special case + Current and present value Hamiltonians

- Our original problem was the maximisation of:

$$\int_0^\infty e^{-\rho t} u\big(t, x(t), c(t)\big)\, dt$$

  - subject to $\dot{x}(t) = f\big(t, x(t), c(t)\big)$.

- This is a special case of the previous general result, with: $\Psi(t, x) = 0$, $\Upsilon(t, x, c, \dot{x}) = e^{-\rho t} u(t, x, c)$, $\Phi(t, x, c, \dot{x}) = f(t, x, c) - \dot{x}$.

- In order to help remember the necessary conditions, it is helpful to define the current and present value "Hamiltonians" $\mathcal{H}_c$ and $\mathcal{H}_p$.
  - Note: the present value Hamiltonian is sometimes just called the Hamiltonian, and denoted $\mathcal{H}$.
  - In macro, the current value one is usually easier to work with.

- These are given by:

$$\mathcal{H}_p(x, c, \mu)(t) = e^{-\rho t} u(t, x, c) + \mu' f(t, x, c),$$

$$\mathcal{H}_c(x, c, \lambda)(t) = e^{\rho t} \mathcal{H}_p(t, x, c, \mu) = u(t, x, c) + \lambda' f(t, x, c),$$

  - where $\lambda(t) = e^{\rho t} \mu(t)$.

Necessary conditions for a maxima in the standard infinite horizon special case

Plain FOCs	FOCs in terms of the present value Hamiltonian	FOCs in terms of the current value Hamiltonian
$e^{-\rho t} u_2(t,x,c) + \mu' f_2(t,x,c) = -\dot{\mu}'$	$\mathcal{H}_{p,1}(x,c,\mu) = -\dot{\mu}'$	$\mathcal{H}_{c,1}(x,c,\lambda) = \rho\lambda' - \dot{\lambda}'$
$e^{-\rho t} u_3(t,x,c) + \mu' f_3(t,x,c) = 0$	$\mathcal{H}_{p,2}(x,c,\mu) = 0$	$\mathcal{H}_{c,2}(x,c,\lambda) = 0$
$\dot{x} = f(t,x,c)$	$\mathcal{H}_{p,3}(x,c,\mu) = \dot{x}'$	$\mathcal{H}_{c,3}(x,c,\lambda) = \dot{x}'$
$\lim_{t\to\infty} \mu(t)' x(t) = 0$	$\lim_{t\to\infty} \mu(t)' x(t) = 0$	$\lim_{t\to\infty} e^{-\rho t} \lambda(t)' x(t) = 0$

where we have used the fact that $\dot{\lambda} = \frac{d}{dt}(e^{\rho t}\mu) = \rho\lambda + e^{\rho t}\dot{\mu}$, so $-e^{\rho t}\dot{\mu} = \rho\lambda - \dot{\lambda}$.

A brief reminder on solving simple differential equations of one variable

- Suppose:

$$\phi(t) = \phi(0) + \int_0^t \psi(s)\, ds\,,$$

- Then $\dot{\phi}(t) = \psi(t)$. So when we have an equation of the form $\dot{\phi}(t) = \psi(t)$, we know the solution is given by $\phi(t) = \phi(0) + \int_0^t \psi(s)\, ds$.

- Now suppose: $e^{\int_0^t \phi(s)\, ds}\psi(t) = \frac{d}{dt}\left[e^{\int_0^t \phi(s)\, ds}\chi(t)\right]$,
- Where: $\frac{d}{dt}\left[e^{\int_0^t \phi(s)\, ds}\chi(t)\right] = \phi(t)e^{\int_0^t \phi(s)\, ds}\chi(t) + e^{\int_0^t \phi(s)\, ds}\dot{\chi}(t)$
- So: $\psi(t) = \phi(t)\chi(t) + \dot{\chi}(t)$.
- Thus if we have an equation of this form, $e^{\int_0^t \phi(s)\, ds}\chi(t) = \chi(0) + \int_0^t e^{\int_0^\tau \phi(s)\, ds}\psi(\tau)\, d\tau$.
- E.g., if $\dot{\chi}(t) + \phi\chi(t) = \psi$, then $e^{\phi t}\chi(t) = \chi(0) + \psi\int_0^t e^{\phi\tau}\, d\tau = \chi(0) + \frac{\psi}{\phi}\left(e^{\phi t} - 1\right)$, so $\chi(t) = e^{-\phi t}\left(\chi(0) - \frac{\psi}{\phi}\right) + \frac{\psi}{\phi}$.

Example: Asset eating

- Specialise further with: $u(x,c)=\log c$, and $f(x,c)=rx-c$, so:
$$\mathcal{H}_c(x,c,\lambda)=\log c+\lambda(rx-c).$$
- At an optimum we have:
$$\mathcal{H}_{c,1}(x,c,\lambda)=\lambda r=\rho\lambda-\dot{\lambda},$$
$$\mathcal{H}_{c,2}(x,c,\lambda)=\frac{1}{c}-\lambda=0,$$
$$\mathcal{H}_{c,3}(x,c,\lambda)=rx-c=\dot{x}.$$
- Hence: $\frac{\dot{\lambda}}{\lambda}=\rho-r$, so $\lambda=\lambda_0 e^{(\rho-r)t}$, $c=\frac{1}{\lambda_0}e^{(r-\rho)t}$ and $\dot{x}=rx-\frac{1}{\lambda_0}e^{(r-\rho)t}$, so $\frac{d}{dt}(e^{-rt}x)=e^{-rt}\dot{x}-re^{-rt}x=re^{-rt}x-\frac{1}{\lambda_0}e^{-\rho t}-re^{-rt}x=-\frac{1}{\lambda_0}e^{-\rho t}$.
- Thus $e^{-rt}x=x_0-\frac{1}{\lambda_0}\int_0^t e^{-\rho\tau}\,d\tau=x_0+\frac{1}{\lambda_0\rho}(e^{-\rho t}-1)$, i.e. $x=e^{rt}\left(x_0+\frac{1}{\lambda_0\rho}\right)-\frac{1}{\lambda_0\rho}e^{(r-\rho)t}$.
- The transversality constraint requires: $0=\lim_{t\to\infty}e^{-\rho t}\lambda(t)x(t)=\lim_{t\to\infty}\lambda_0\left[x_0+\frac{1}{\lambda_0\rho}-\frac{1}{\lambda_0\rho}e^{-\rho t}\right]$.
- Thus assuming a borrowing constraint, $\lambda_0=-\frac{1}{\rho x_0}$, so $x=x_0e^{(r-\rho)t}$, and $c=\rho x_0e^{(r-\rho)t}=\rho x$.

Example: Non-renewable resource extraction by a monopolist (Hotelling 1931)

- A monopolist controls the stock of some non-renewable resource.
  - Let $x(t)$ be the amount of the resource they have remaining at t.
  - Assume that the unit cost of resource extraction is given by $c(t)$.
  - Assume demand at a price p is given by $q(t,p) = a(t)p^{-\kappa}$.
- The monopolist maximises the present value of profits, given a constant real interest rate r, namely:

$$\Pi(x,p) = \int_0^\infty e^{-rt} q\big(t, p(t)\big)[p(t) - c(t)]\, dt\,,$$

  - subject to the resource constraint: $\dot{x}(t) = -q\big(t, p(t)\big)$.
- The current value Hamiltonian is:

$$\mathcal{H}_c(x,p,\lambda) = ap^{-\kappa}[p - c] - \lambda a p^{-\kappa}$$

- At an optimum, we have:

$$\mathcal{H}_{c,1}(x,p,\lambda) = 0 = r\lambda - \dot{\lambda},$$
$$\mathcal{H}_{c,2}(x,p,\lambda) = -\kappa a p^{-\kappa-1}(p - c) + ap^{-\kappa} + \lambda\kappa a p^{-\kappa-1} = 0,$$
$$\mathcal{H}_{c,3}(x,p,\lambda) = -ap^{-\kappa} = \dot{x}.$$

- From the first equation, $\lambda(t) = \lambda_0 e^{rt}$, so from the second: $p(t) = \frac{\kappa}{\kappa-1}\big(c(t) + \lambda_0 e^{rt}\big)$.

Example: Ramsey-Cass-Koopmans model of exogenous growth (1/3)

- The production function in an economy is given by: $Y = K^{\alpha}(AL)^{1-\alpha}$,
  - where $L = L_0 e^{nt}$ is population, $A = A_0 e^{gt}$ is productivity and capital K evolves according to $\dot{K} = Y - C - \delta K$, where C is consumption.
- Let lower case variables be per efficiency unit equivalents (i.e. the original divided by AL).
- Then $y = k^{\alpha}$, and $\dot{k} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{K}{AL}\right) = \frac{\dot{K}AL - K(\dot{A}L + A\dot{L})}{(AL)^2} = \frac{Y - C - \delta K}{AL} - \frac{K}{AL}\left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L}\right) = y - c - (\delta + g + n)k$.
- Note consumption per head is $\frac{C}{L} = cA_0 e^{gt}$. So, the social planner chooses k and c to maximise:

$$V(k,c) = \int_0^{\infty} e^{-\rho t} \frac{(cA_0 e^{gt})^{1-\sigma} - 1}{1-\sigma} dt = -\frac{1}{\rho(1-\sigma)} + A_0^{1-\sigma} \int_0^{\infty} e^{-\nu t} \frac{c^{1-\sigma}}{1-\sigma} dt\,,$$

  - where $\nu \coloneqq \rho - g(1-\sigma)$,
  - subject to $\dot{k} = k^{\alpha} - c - (\delta + g + n)k$.

Example: Ramsey-Cass-Koopmans model of exogenous growth (2/3)

- The current value Hamiltonian for the problem is:

$$\mathcal{H}_c(k, c, \lambda) = \frac{c^{1-\sigma}}{1-\sigma} + \lambda(k^\alpha - c - (\delta + g + n)k).$$

- At an optimum we have:

$$\mathcal{H}_{c,1}(k, c, \lambda) = \lambda[\alpha k^{\alpha-1} - (\delta + g + n)] = \nu\lambda - \dot{\lambda},$$
$$\mathcal{H}_{c,2}(k, c, \lambda) = c^{-\sigma} - \lambda = 0,$$
$$\mathcal{H}_{c,3}(k, c, \lambda) = k^\alpha - c - (\delta + g + n)k = \dot{k}.$$

- From the second equation, $\dot{\lambda} = -\sigma\, c^{-\sigma-1}\dot{c}$, hence, from the first, we have the "Euler" equation:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}[\alpha k^{\alpha-1} - (\delta + g + n) - \nu] = \frac{1}{\sigma}[\alpha k^{\alpha-1} - (\delta + \sigma g + n) - \rho]$$

Example: Ramsey-Cass-Koopmans model of exogenous growth (3/3)

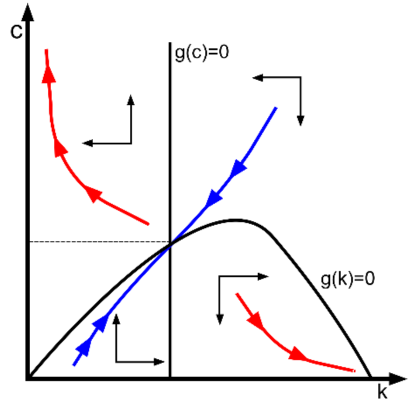
- In "steady-state" $\dot{c} = \dot{k} = 0$. Note:

$$\dot{c} = 0 \Rightarrow k = \left(\frac{\alpha}{\delta + \sigma g + n + \rho}\right)^{\frac{1}{1-\alpha}},$$
$$\dot{k} = 0 \Rightarrow c = k^{\alpha} - (\delta + g + n)k.$$

- Transversality implies:

$$\lim_{t \to \infty} e^{-\nu t} c(t)^{-\sigma} k(t) = 0.$$

- This rules out all paths except the blue one on the phase diagram below:



Source: https://en.wikipedia.org/wiki/File:Ramseypic.svg

Techniques for solving systems of nonlinear ordinary differential equations (ODEs)

- Suppose $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$.
- The steady-state x^* solves $0 = f(x)$.
- To a first order approximation: $f(x) \approx f_1(x^*)(x - x^*)$, where $f_1(x^*)$ is the Jacobian of f evaluated at x^*.
- If $f_1(x^*)$ is diagonalizable, it is then very easy to get an approximate solution.
  - Write $f_1(x^*) = VDV^{-1}$, and define $y = V^{-1}(x - x^*)$. Then $\dot{y} = V^{-1}\dot{x} = V^{-1}f(x) \approx V^{-1}f_1(x^*)(x - x^*) = V^{-1}VDV^{-1}(x - x^*) = Dy$.
  - Hence, $y_i(t) = y_{i,0}e^{D_{ii}t}$, which gives x from $x = Vy + x^*$.

- However, unlike with discrete time models, it's also quite easy to solve the system fully nonlinearly.
  - A crude algorithm (the Euler method) discretises time and treats the model as $\Delta x_t = hf(x_{t-h})$, where h is the time step. This is based on a first order approximation to the derivative.
  - More accurate approximations to the derivative deliver more accurate measures.
  - MATLAB contains many different ODE solvers. ode45 is a good starting point.

Hamilton-Jacobi-Bellman (HJB) equations: Finite horizon case

- Just as in discrete time, we can also tackle optimal control problems via a Bellman equation approach.
- Suppose:

$$\mathcal{V}\big(t, x(t)\big) = \max_c \left[\int_t^T \Upsilon\big(\tau, x(\tau), c(\tau)\big)\, d\tau + \Psi\big(x(T)\big)\right]$$

  - subject to the constraint that $\dot{x}(t) = \Phi\big(t, x(t), c(t)\big)$.
- Then, for small (infinitesimal) dt:

$$\mathcal{V}(t, x) = \max_c [\Upsilon(t, x, c)\, dt + \mathcal{V}(t + dt, x + \Phi(t, x, c)\, dt)]$$

- I.e.:

$$0 = \max_c \left[\Upsilon(t, x, c) + \frac{\mathcal{V}(t + dt, x + \Phi(t, x, c)\, dt) - \mathcal{V}(t, x)}{dt}\right]$$

- Hence (or at least by this intuition), the HJB partial differential equation (PDE) is:

$$-\mathcal{V}_1(t, x) = \max_c [\Upsilon(t, x, c) + \mathcal{V}_2(t, x)\Phi(t, x, c)],$$

  - which must be solved subject to the terminal condition $\mathcal{V}(T, x) = \Psi\big(x(T)\big)$.
  - c will satisfy the standard FOC: $\Upsilon_3(t, x, c) + \mathcal{V}_2(t, x)\Phi_3(t, x, c) = 0$.

- Whereas the previous method, based on Euler-Lagrange equations, gave necessary conditions for optimality, the HJB equation gives necessary and sufficient conditions, when solved globally.

HJB equations: Infinite horizon case

- Suppose:

$$\mathcal{V}\big(t, x(t)\big) = \max_{c}\left[\int_t^{\infty} e^{-\rho\tau} u\big(x(\tau), c(\tau)\big)\, d\tau\right]$$

  - subject to the constraint that $\dot{x}(t) = f\big(x(t), c(t)\big)$.
- By the same steps as before, this gives an HJB equation of the form:

$$-\mathcal{V}_1(t, x) = \max_{c}[e^{-\rho t} u(x, c) + \mathcal{V}_2(t, x) f(x, c)]$$

- We then make the informed guess that $\mathcal{V}(t, x) = e^{-\rho t} V(x)$.
  - This implies that $\mathcal{V}_1(t, x) = -\rho\mathcal{V}(t, x)$, and that $\mathcal{V}_2(t, x) = e^{-\rho t} V_1(x)$.
- Hence:

$$\rho V(x) = \max_{c}[u(x, c) + V_1(x) f(x, c)],$$

  - where $\mathcal{V}(t, x) = e^{-\rho t} V(x)$.
  - c will satisfy the standard FOC: $u_2(x, c) + V_1(x) f_2(x, c) = 0$.

Link to Hamiltonians

- Recall that $\mathcal{H}_c(x, c, \lambda) = u(x, c) + \lambda' f(x, c)$.
- The FOC for c is connected to our previous current-value Hamiltonian method through the substitution $\lambda' = V_1(x)$, since:

$$0 = \mathcal{H}_{c,2}(x, c, \lambda) = u_2(x, c) + \lambda' f_2(x, c) = u_2(x, c) + V_1(x) f_2(x, c).$$

- Thus the HJB equation is just:

$$\rho V(x) = \max_c \mathcal{H}_c(x, c, V_1(x)').$$

- Differentiating with respect to x gives (using the envelope theorem):

$$\begin{aligned} \rho V_1(x) &= \mathcal{H}_{c,1}(x, c, V_1(x)') + \mathcal{H}_{c,3}(x, c, V_1(x)') V_{1'1}(x) \\ &= \mathcal{H}_{c,1}(x, c, V_1(x)') + f(x, c)' V_{1'1}(x) \end{aligned}$$

- Now: $\dot{\lambda} = \frac{dV_1(x)'}{dt} = V_{1'1}(x)\dot{x} = V_{1'1}(x) f(x, c)$.
- Thus: $\rho\lambda' - \dot{\lambda}' = \mathcal{H}_{c,1}(x, c, V_1(x)')$.

Solving HJB equations

- Global numerical techniques proceed (as in discrete time) by approximating the value function over a grid.
- For some very simple models, analytic solutions may be derived by solving the PDE.
- For moderately simple models, analytic solutions may be derived via a "guess and verify" approach.

- For example, consider again the asset eating problem with $u(x, c) = \log c$, and $f(x, c) = rx - c$.
  - Then the HJB equation is: $\rho V(x) = \max_c [\log c + V_1(x)(rx - c)]$.
  - Informed guess: $V(x) = a \log(bx)$. (Implicitly imposing borrowing constraint.)
  - Then the FOC for c gives $\frac{1}{c} = \frac{a}{x}$, so $c = \frac{x}{a}$.
  - Substituting in, we have $\rho a \log b + \rho a \log x = \log x - \log a + ar - 1$, so clearly $a = \frac{1}{\rho}$ (so $c = \rho x$ as before) and: $b = \rho \exp\left(\frac{r}{\rho} - 1\right)$.

Multivariate Ito's lemma

- Suppose:

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t$$

- where $X_t, \mu_t \in \mathbb{R}^n$, $\sigma_t \in \mathbb{R}^{n\times m}$ and W_t is an m dimensional vector of independent Brownian motions.

- Then, if $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$:

$$df(t, X_t) = \left(f_1(t, X_t) + f_2(t, X_t)\mu_t + \frac{1}{2}\operatorname{tr}(\sigma_t' f_{2'2}(t, X_t)\sigma_t) \right) dt + f_2(t, X_t)\sigma_t \, dW_t,$$

- where $f_{2'2}(t, X_t)$ is the Hessian of f with respect to its second argument.

Stochastic HJB equations

- We just show the infinite horizon case here. Suppose:

$$\mathcal{V}\big(t, x(t)\big) = \max_c \mathbb{E}_t \left[\int_t^\infty e^{-\rho\tau} u\big(x(\tau), c(\tau)\big)\, d\tau\right]$$

  - subject to the constraint that $dx(t) = f\big(x(t), c(t)\big)\, dt + \sigma\big(x(t), c(t)\big)\, dW(t)$.

- Then the (non-stochastic!) HJB equation is:

$$\rho V(x) = \max_c \left[u(x, c) + V_1(x) f(x, c) + \frac{1}{2}\operatorname{tr}\big(\sigma(x, c)' V_{1'1}(x) \sigma(x, c)\big)\right],$$

  - where $\mathcal{V}(t, x) = e^{-\rho t} V(x)$, where $V_{1'1}(x)$ is the Hessian of V.
  - You will recognise the final term from Ito's lemma.
  - c will satisfy the standard FOC:

$$u_2(x, c) + V_1(x) f_2(x, c) + \operatorname{vec}\left(V_{1'1}(x)^{\frac{1}{2}} \sigma(x, c)\right)' \left(I \otimes V_{1'1}(x)^{\frac{1}{2}}\right) \frac{\partial \operatorname{vec} \sigma(x, c)}{\partial c} = 0,$$

  - where we have used the fact that:

$$\operatorname{tr}\big(\sigma(x, c)' V_{1'1}(x) \sigma(x, c)\big) = \operatorname{tr}\left[\left(V_{1'1}(x)^{\frac{1}{2}} \sigma(x, c)\right)' \left(V_{1'1}(x)^{\frac{1}{2}} \sigma(x, c)\right)\right] = \operatorname{vec}\left(V_{1'1}(x)^{\frac{1}{2}} \sigma(x, c)\right)' \operatorname{vec}\left(V_{1'1}(x)^{\frac{1}{2}} \sigma(x, c)\right).$$

- The remarkable thing is that the stochastic system has been converted to a non-stochastic set of PDEs.

Applications in macroeconomics of the stochastic HJB

- Olaf Posch has pioneered the recent application of the stochastic HJB in macroeconomics, building on earlier work by Merton.

- Richer models may be solved analytically in continuous time.

- Non-linearities are far easier to handle in continuous time.
  - See e.g. Fernandez-Villaverde, Posch and Rubio-Ramirez (2012) who solve an NK model with the ZLB in continuous time, getting analytic results for a special case, and accurate numerical results more generally.
  - Or Posch (2010) which derives analytic expression for general equilibrium risk premia, based on an extension of the Merton (1975) model.

Example: Stochastic asset eating without short selling constraints (1/2)

- Suppose a household maximises the Lagrangian:

$$\mathcal{V}\big(t, x(t)\big) = \max_{c,z,\mu} \mathbb{E}_t \left[\int_t^\infty e^{-\rho\tau} \left[\frac{c(\tau)^{1-\sigma} - 1}{1-\sigma} - \mu(\tau)(1_n' z(\tau) - 1) \right] d\tau \right]$$

  - subject to the constraints that $dx(t) = \big(x(t)z(t)'r - c(t)\big)\,dt + x(t)z(t)'\Lambda\, dW(t)$.
  - $x(t)z(t)$ gives the vector of asset holdings at t, $x(t)$ is total net worth.

- Stochastic HJB equation is:

$$\begin{aligned} \rho V(x) &= \max_{c,z,\mu} \left[\frac{c^{1-\sigma} - 1}{1-\sigma} - \mu(1_n' z - 1) + V_1(x)(xz'r - c) + \frac{1}{2}\operatorname{tr}(\Lambda' z x V_{11}(x) x z' \Lambda) \right] \\ &= \max_{c,z,\mu} \left[\frac{c^{1-\sigma} - 1}{1-\sigma} - \mu(1_n' z - 1) + V_1(x)(xz'r - c) + \frac{1}{2} V_{11}(x) x^2 z' \Lambda\Lambda' z \right]. \end{aligned}$$

- FOCs:

$$c^{-\sigma} = V_1(x), \qquad \Rightarrow c = V_1(x)^{-\frac{1}{\sigma}},$$

$$V_1(x) x r' + V_{11}(x) x^2 z' \Lambda\Lambda' = \mu 1_n', \qquad \Rightarrow z = \frac{(\Lambda\Lambda')^{-1}(\mu 1_n - V_1(x) x r)}{V_{11}(x) x^2},$$

$$1_n' z = 1, \Rightarrow \mu = \frac{V_1(x) x 1_n' (\Lambda\Lambda')^{-1} r + V_{11}(x) x^2}{1_n' (\Lambda\Lambda')^{-1} 1_n}, \Rightarrow z = \left(\frac{V_1(x) x 1_n' (\Lambda\Lambda')^{-1} r + V_{11}(x) x^2}{1_n' (\Lambda\Lambda')^{-1} 1_n} \right) \frac{(\Lambda\Lambda')^{-1} 1_n}{V_{11}(x) x^2} - \frac{V_1(x) x (\Lambda\Lambda')^{-1} r}{V_{11}(x) x^2},$$

Example: Stochastic asset eating without short selling constraints (2/2)

- Define $\langle u,v\rangle \coloneqq u'(\Lambda\Lambda')^{-1}v$ then plugging in and simplifying gives:

$$\rho V(x)=\frac{V_1(x)^{-\frac{1-\sigma}{\sigma}}-1}{1-\sigma}-V_1(x)V_1(x)^{-\frac{1}{\sigma}}+\frac{1}{2}\frac{(V_1(x)x\langle 1_n,r\rangle+V_{11}(x)x^2)^2-(V_1(x)x)^2\langle 1_n,1_n\rangle\langle r,r\rangle}{V_{11}(x)x^2\langle 1_n,1_n\rangle}.$$

- Guess $V(x)=a\frac{(bx)^{1-\sigma}-1}{1-\sigma}$, so $V_1(x)=ab(bx)^{-\sigma}$ and $V_{11}(x)=-\sigma ab^2(bx)^{-\sigma-1}$, so after simplifying we have:

$$\frac{\rho a}{1-\sigma}(bx)^{1-\sigma}-\rho a\frac{1}{1-\sigma}=\left[\frac{\sigma}{1-\sigma}(ab)^{-\frac{1-\sigma}{\sigma}}+\frac{1}{2}a\frac{\langle 1_n,1_n\rangle\langle r,r\rangle-(\langle 1_n,r\rangle-\sigma)^2}{\sigma\langle 1_n,1_n\rangle}\right](bx)^{1-\sigma}-\frac{1}{1-\sigma}.$$

- Thus $a=\frac{1}{\rho}$ (as in the non-stochastic case) and

$$b=\rho\left[\frac{1}{\sigma}-\frac{1-\sigma}{\sigma}\frac{\langle 1_n,1_n\rangle\langle r,r\rangle-(\langle 1_n,r\rangle-\sigma)^2}{2\sigma\rho\langle 1_n,1_n\rangle}\right]^{-\frac{\sigma}{1-\sigma}}.$$

- As $\sigma\to 1$, $b\to\rho\exp\left[\frac{\langle 1_n,1_n\rangle\langle r,r\rangle-(\langle 1_n,r\rangle-1)^2}{2\rho\langle 1_n,1_n\rangle}-1\right]$.

- In the scalar ($n=1$) case, $\frac{\langle 1_n,1_n\rangle\langle r,r\rangle-(\langle 1_n,r\rangle-\sigma)^2}{2\sigma\rho\langle 1_n,1_n\rangle}=\frac{r-\frac{1}{2}\sigma\Lambda^2}{\rho}$, which implies the agent values assets less the more risk averse they are, and the more risky is the asset.