Continuous time optimal control

Tom Holden

http://www.tholden.org/

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Outline of today’s talk

• Continuous time optimisation problems.
• Euler-Lagrange Equations.
• Necessary conditions for general continuous time optimal control.
• The standard infinite horizon case, and the current and present value Hamiltonians.
• Assorted examples.
• Hamilton-Jacobi-Bellman (HJB) equations.
• Stochastic HJB equations.
Reading for today

• Appendices of any growth textbook. E.g.:
  • “Economic Growth”: Barro and Sala-i-Martin
  • “The Economics of Growth”: Aghion and Howitt
  • “Introduction to Modern Economic Growth”: Acemoglu

• “The Economics of Inaction: Stochastic Control Models with Fixed Costs”: Stokey (2008)
  • For the stochastic HJB material.

• “Solving the New Keynesian Model in Continuous Time”: Fernandez-Villaverde, Posch and Rubio-Ramirez (2012)
  • Great, readable example of what is possible in continuous time.
Example continuous time optimisation problem

• In continuous time macro, we are often interested in finding the paths for control variables $c$ and state variables $x$ that maximise objectives of the form:

$$\int_0^\infty e^{-\rho t} u(x(t), c(t)) \, dt$$

• given the initial value of the state, $x(0) = x_0$, where the state evolves according to:

$$\dot{x}(t) = f(x(t), c(t)),$$

• and where $\rho$ is the discount rate.
  • Note: throughout this lecture dots above variables will denote derivatives with respect to time.

• For example, in a simple asset eating model, we might have, $u(x, c) = \log c$, and $f(x, c) = rx - c$, where $r$ is the real interest rate.
The Euler-Lagrange Equation

• Suppose we seek to find the stationary points of the functional $S: \mathbb{R} \to \mathbb{R}^n \to \mathbb{R}$ given by:

$$S(z) = \int_0^T \mathcal{L}(t, z(t), \dot{z}(t)) \, dt.$$  

  • where $\mathcal{L}$ has continuous first partial derivatives.

• Note:
  • A functional is a scalar valued function of a function, in this case the function in question $z: \mathbb{R} \to \mathbb{R}^n$.
  • Any maximum or minimum of $S$ must also be a stationary point of it.

• The Euler-Lagrange equation states that if $z$ is a stationary point of $S$, then for all $i \in 1, \ldots, n$, and all $t \in (0, T)$:

$$\frac{\partial \mathcal{L}(t, z(t), \dot{z}(t))}{\partial z_i(t)} = \frac{d}{dt} \frac{\partial \mathcal{L}(t, z(t), \dot{z}(t))}{\partial \dot{z}_i(t)}.$$  

• Or in more compact notation: $\mathcal{L}_2(t, z(t), \dot{z}(t)) = \frac{d}{dt} \mathcal{L}_3(t, z(t), \dot{z}(t))$,

  • where, for a column vector valued function $f(x, y)$, and e.g. $y \in \mathbb{R}^n$, $f_2(x, y)$ is the matrix

$$\begin{bmatrix}
\frac{\partial f(x, y)}{\partial y_1} & \ldots & \frac{\partial f(x, y)}{\partial y_n}
\end{bmatrix}.$$
Transversality constraints

• Unless the terminal value of the state is specified in advance, then an additional condition is needed in order to pin down the path of the state and costate.
  • An example of a situation in which the terminal state is specified would be seeking the optimal rate at which to collect a given stock of assets.

• The necessary condition for optimality is that:
  \[ \mathcal{L}_3(T, z(T), \dot{z}(T)) = 0. \]

• Intuitively, this states that an extra unit of the state is worthless at the end of time.
  • This result is often combined with the Euler-Lagrange equation result under the banner of “Pontryagin’s Maximum Principle”.

• We view infinite horizon models as the limit of finite horizon ones as \( T \to \infty \), thus in the infinite horizon case you might think that a necessary condition would be \( \lim_{T \to \infty} \mathcal{L}_3(T, z(T), \dot{z}(T)) = 0 \). In fact, this is not necessary.

• Kamihigashi (2001) shows that under weak assumptions the necessary transversality condition is:
  \[ \lim_{T \to \infty} \mathcal{L}_3(T, z(T), \dot{z}(T))z(T) = 0 \]
General continuous time optimal control (1/2)

• Suppose we seek to maximise:

\[
\Gamma(x, c) = \Psi(T, x(T)) + \int_0^T \gamma(t, x(t), c(t), \dot{x}(t)) \, dt,
\]

subject to: \(0 = \Phi(t, x(t), c(t), \dot{x}(t))\) and \(x(0) = x_0\).

• First note that:

\[
\frac{d}{dt}\Psi(t, x(t)) = \Psi_1(t, x(t)) + \Psi_2(t, x(t))\dot{x}(t).
\]

• So:

\[
\int_0^T \left[\Psi_1(t, x(t)) + \Psi_2(t, x(t))\dot{x}(t)\right] \, dt = \Psi(T, x(T)) - \Psi(0, x_0).
\]

• Hence:

\[
\Gamma(x, c) = \Psi(0, x_0) + \int_0^T \left[\gamma(t, x(t), c(t), \dot{x}(t)) + \Psi_1(t, x(t)) + \Psi_2(t, x(t))\dot{x}(t)\right] \, dt.
\]

• We proceed by applying the previous result to the Lagrangian:

\[
\mathcal{L}(t, z, \dot{z}) = \gamma(t, x, c, \dot{x}) + \Psi_1(t, x) + \Psi_2(t, x)\dot{x} + \mu'\Phi(t, x, c, \dot{x}),
\]

where \(z = [x', c', \mu']\).
General continuous time optimal control (2/2)

• From the previous result, the stationary points of $S(z) = \int_0^T \mathcal{L}(t, z(t), \dot{z}(t)) \, dt$ satisfy:

$$\Upsilon_2(t, x, c, \dot{x}) + \Psi_{12}(t, x(t)) + \Psi_{22}(t, x(t)) \dot{x}(t) + \mu' \Phi_2(t, x, c, \dot{x})$$

$$= \frac{d}{dt} \left( \Upsilon_4(t, x, c, \dot{x}) + \Psi_2(t, x(t)) + \mu' \Phi_4(t, x, c, \dot{x}) \right),$$

$$\Upsilon_3(t, x, c, \dot{x}) + \mu' \Phi_3(t, x, c, \dot{x}) = 0,$$

$$\Phi(t, x, c, \dot{x}) = 0,$$

$$\Upsilon_4(T, x(T), c(T), \dot{x}(T)) + \Psi_2(T, x(T)) + \mu'(T) \Phi_4(T, x(T), c(T), \dot{x}(T)) = 0.$$  

• Note that the penultimate equation implies that at a stationary point of $S$, the constraint is satisfied, as required.

• Also, note that the first equation simplifies to:

$$\Upsilon_2(t, x, c, \dot{x}) + \mu' \Phi_2(t, x, c, \dot{x}) = \frac{d}{dt} \left( \Upsilon_4(t, x, c, \dot{x}) + \mu' \Phi_4(t, x, c, \dot{x}) \right),$$

• as $\Psi_{21}(t, x) = \Psi_{12}(t, x)$. 
A standard infinite horizon special case + Current and present value Hamiltonians

• Our original problem was the maximisation of:
  \[ \int_0^\infty e^{-\rho t} u(x(t), c(t)) \, dt \]
  • subject to \( \dot{x}(t) = f(x(t), c(t)) \).
• This is a special case of the previous general result, with: \( \Psi(t, x) = 0, \)
  \( \Upsilon(t, x, c, \dot{x}) = e^{-\rho t} u(x, c), \Phi(t, x, c, \dot{x}) = f(x, c) - \dot{x} \).
• In order to help remember the necessary conditions, it is helpful to define
  the current and present value “Hamiltonians” \( \mathcal{H}_c \) and \( \mathcal{H}_p \).
  • Note: the present value Hamiltonian is sometimes just called the Hamiltonian,
    and denoted \( \mathcal{H} \).
  • In macro, the current value one is usually easier to work with.
• These are given by:
  \[ \mathcal{H}_p(x, c, \mu) = L(t, z, \dot{z}) + \mu' \dot{x} = e^{-\rho t} u(x, c) + \mu' f(x, c), \]
  \[ \mathcal{H}_c(x, c, \lambda) = e^{\rho t} \mathcal{H}_p(x, c, \mu) = u(x, c) + \lambda' f(x, c), \]
  • where \( \lambda(t) = e^{\rho t} \mu(t) \).
Necessary conditions for a maxima in the standard infinite horizon special case

<table>
<thead>
<tr>
<th>Plain FOCs</th>
<th>FOCs in terms of the present value Hamiltonian</th>
<th>FOCs in terms of the current value Hamiltonian</th>
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<tbody>
<tr>
<td>$e^{-\rho t}u_1(x, c) + \mu' f_1(x, c) = -\dot{\mu}'$</td>
<td>$\mathcal{H}_{p,1}(x, c, \mu) = -\dot{\mu}'$</td>
<td>$\mathcal{H}_{c,1}(x, c, \lambda) = \rho \lambda' - \dot{\lambda}'$</td>
</tr>
<tr>
<td>$e^{-\rho t}u_2(x, c) + \mu' f_2(x, c) = 0$</td>
<td>$\mathcal{H}_{p,2}(x, c, \mu) = 0$</td>
<td>$\mathcal{H}_{c,2}(x, c, \lambda) = 0$</td>
</tr>
<tr>
<td>$\dot{x}(t) = f(x(t), c(t))$</td>
<td>$\mathcal{H}_{p,3}(x, c, \mu) = \dot{x}'$</td>
<td>$\mathcal{H}_{c,3}(x, c, \lambda) = \dot{x}'$</td>
</tr>
<tr>
<td>$\lim_{t \to \infty} \mu(t)' x(t) = 0$</td>
<td>$\lim_{t \to \infty} \mu(t)' x(t) = 0$</td>
<td>$\lim_{t \to \infty} e^{-\rho t} \lambda(t)' x(t) = 0$</td>
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where we have used the fact that $\dot{\lambda} = \frac{d}{dt} (e^{\rho t} \mu) = \rho \lambda + e^{\rho t} \dot{\mu}$, so $-e^{\rho t} \dot{\mu} = \rho \lambda - \dot{\lambda}$. 
A brief reminder on solving simple differential equations of one variable

• Suppose:
  \[ \phi(t) = \phi(0) + \int_0^t \psi(s) \, ds, \]

• Then \( \dot{\phi}(t) = \psi(t) \). So when we have an equation of the form \( \dot{\phi}(t) = \psi(t) \), we know the solution is given by \( \phi(t) = \phi(0) + \int_0^t \psi(s) \, ds \).

• Now suppose:
  \[ e^{\int_0^t \phi(s) \, ds} \psi(t) = \frac{d}{dt} \left[ e^{\int_0^t \phi(s) \, ds} \chi(t) \right] = \phi(t) e^{\int_0^t \phi(s) \, ds} \chi(t) + e^{\int_0^t \phi(s) \, ds} \dot{\chi}(t). \]

• So: \( \psi(t) = \phi(t) \chi(t) + \dot{\chi}(t) \).

• Thus if we have an equation of this form, \( e^{\int_0^t \phi(s) \, ds} \chi(t) = \chi(0) + \int_0^t e^{\int_0^\tau \phi(s) \, ds} \psi(\tau) \, d\tau \).

• Hence, for example, if \( \dot{\chi}(t) + \phi \chi(t) = \psi \), then \( e^{\phi t} \chi(t) = \chi(0) + \psi \int_0^t e^{\phi \tau} \, d\tau = \chi(0) + \frac{\psi}{\phi} (e^{\phi t} - 1) \), so \( \chi(t) = e^{-\phi t} \left( \chi(0) - \frac{\psi}{\phi} \right) + \frac{\psi}{\phi} \).
Example: Asset eating

• Specialise further with: $u(x, c) = \log c$, and $f(x, c) = rx - c$, so:
  $$\mathcal{H}_c(x, c, \lambda) = \log c + \lambda(rx - c).$$

• At an optimum we have:
  $$\mathcal{H}_{c,1}(x, c, \lambda) = \lambda r = \rho \lambda - \dot{\lambda},$$
  $$\mathcal{H}_{c,2}(x, c, \lambda) = \frac{1}{c} - \lambda = 0,$$
  $$\mathcal{H}_{c,3}(x, c, \lambda) = rx - c = \dot{x}.$$

• Hence: $\frac{\dot{\lambda}}{\lambda} = \rho - r$, so $\lambda = \lambda_0 e^{(\rho-r)t}$, $c = \frac{1}{\lambda_0} e^{(r-\rho)t}$ and $\dot{x} = rx - \frac{1}{\lambda_0} e^{(r-\rho)t}$, so $\frac{d}{dt}(e^{-rt}x) = e^{-rt} \dot{x} - re^{-rt}x = re^{-rt}x - \frac{1}{\lambda_0} e^{-\rho t} - re^{-rt}x = -\frac{1}{\lambda_0} e^{-\rho t}$.

• Thus $e^{-rt}x = x_0 - \frac{1}{\lambda_0} \int_0^t e^{-\rho \tau} d\tau = x_0 - \frac{1}{\lambda_0 \rho} (e^{-\rho t} - 1)$, i.e. $x = e^{rt} \left(x_0 - \frac{1}{\lambda_0 \rho}\right) + \frac{1}{\lambda_0 \rho} e^{(r-\rho)t}$.

• The transversality constraint requires: $0 = \lim_{t \to \infty} e^{-\rho t} \lambda(t)x(t) = \lim_{t \to \infty} \lambda_0 \left[x_0 - \frac{1}{\lambda_0 \rho} + \frac{1}{\lambda_0 \rho} e^{-\rho t}\right]$.

• Thus $\lambda_0 = \frac{1}{\rho x_0}$, so $x = x_0 e^{(r-\rho)t}$, and $c = \rho x_0 e^{(r-\rho)t} = \rho x$. 
Example: Non-renewable resource extraction by a monopolist (Hotelling 1931)

• A monopolist controls the stock of some non-renewable resource.
  • Let $x(t)$ be the amount of the resource they have remaining at $t$.
  • Assume that the unit cost of resource extraction is given by $c(t)$.
  • Assume demand at a price $p$ is given by $q(t, p) = a(t)p^{-\kappa}$.

• The monopolist maximises the present value of profits, given a constant real interest rate $r$, namely:
  $$\Pi(x, p) = \int_0^\infty e^{-rt} q(t, p(t))[p(t) - c(t)] \, dt,$$

  • subject to the resource constraint: $\dot{x}(t) = -q(t, p(t))$.

• The current value Hamiltonian is:
  $$H_c(x, p, \lambda) = ap^{-\kappa}[p - c] - \lambda ap^{-\kappa}$$

• At an optimum, we have:
  $$H_{c,1}(x, p, \lambda) = 0 = r\lambda - \dot{\lambda},$$
  $$H_{c,2}(x, p, \lambda) = -\kappa ap^{-\kappa-1}(p - c) + ap^{-\kappa} + \lambda kap^{-\kappa-1} = 0,$$
  $$H_{c,3}(x, q, \lambda) = -ap^{-\kappa} = \dot{x}.$$

• From the first equation, $\lambda(t) = \lambda_0 e^{rt}$, so from the second: $p(t) = \frac{\kappa}{\kappa-1} (c(t) + \lambda_0 e^{rt})$. 

Example: Ramsey-Cass-Koopmans model of exogenous growth (1/3)

• The production function in an economy is given by: \( Y = K^\alpha (AL)^{1-\alpha} \),
  • where \( L = L_0 e^{nt} \) is population, \( A = A_0 e^{gt} \) is productivity and capital \( K \) evolves according to \( \dot{K} = Y - C - \delta K \), where \( C \) is consumption.

• Let lower case variables be per efficiency unit equivalents (i.e. the original divided by \( AL \)).

• Then \( y = k^\alpha \), and \( \dot{k} = \frac{d}{dt} \left( \frac{K}{AL} \right) = \frac{K(AL-K(\dot{A}L+\dot{AL}))}{(AL)^2} = \frac{Y-C-\delta K}{AL} - \frac{K}{AL} \left( \frac{\dot{A}}{A} + \frac{\dot{L}}{L} \right) = y - c - (\delta + g + n)k \).

• Note consumption per head is \( \frac{C}{L} = cA_0 e^{gt} \). So, the social planner chooses \( k \) and \( c \) to maximise:

\[
V(k, c) = \int_0^\infty e^{-\rho t} \left( \frac{cA_0 e^{gt}}{1-\sigma} \right)^{1-\sigma} - 1 \, dt = -\frac{1}{\rho(1-\sigma)} + A_0^{1-\sigma} \int_0^\infty e^{-vt} \frac{c^{1-\sigma}}{1-\sigma} \, dt,
\]

• where \( v := \rho - g(1-\sigma) \),

• subject to \( \dot{k} = k^\alpha - c - (\delta + g + n)k \).
Example: Ramsey-Cass-Koopmans model of exogenous growth (2/3)

• The current value Hamiltonian for the problem is:

\[
H_c(k, c, \lambda) = \frac{c^{1-\sigma}}{1-\sigma} + \lambda(k^\alpha - c - (\delta + g + n)k).
\]

• At an optimum we have:

\[
H_{c,1}(k, c, \lambda) = \lambda[\alpha k^{\alpha-1} - (\delta + g + n)] = v\lambda - \dot{\lambda},
\]

\[
H_{c,2}(k, c, \lambda) = c^{-\sigma} - \lambda = 0,
\]

\[
H_{c,3}(k, c, \lambda) = k^\alpha - c - (\delta + g + n)k = \dot{k}.
\]

• From the second equation, \( \dot{\lambda} = -\sigma c^{-\sigma-1} \dot{c} \), hence, from the first, we have the "Euler" equation:

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} [\alpha k^{\alpha-1} - (\delta + g + n) - v] = \frac{1}{\sigma} [\alpha k^{\alpha-1} - (\delta + \sigma g + n) - \rho]
\]
Example: Ramsey-Cass-Koopmans model of exogenous growth (3/3)

• In “steady-state” $\dot{c} = \dot{k} = 0$. Note:

$$
\dot{c} = 0 \Rightarrow k = \left( \frac{\alpha}{\delta + \sigma g + n + \rho} \right)^{\frac{1}{1-\alpha}},
$$

$$
\dot{k} = 0 \Rightarrow c = k^\alpha - (\delta + g + n)k.
$$

• Transversality implies:

$$
\lim_{t \to \infty} e^{-\nu t} c(t)^{-\sigma} k(t) = 0.
$$

• This rules out all paths except the blue one on the phase diagram below:

Source: https://en.wikipedia.org/wiki/File:Ramseypic.svg
Techniques for solving systems of nonlinear ordinary differential equations (ODEs)

• Suppose \( \dot{x} = f(x) \), where \( x \in \mathbb{R}^n \).

• The steady-state \( x^* \) solves \( 0 = f(x) \).

• To a first order approximation: \( f(x) \approx f_1(x^*)(x - x^*) \), where \( f_1(x^*) \) is the Jacobian of \( f \) evaluated at \( x^* \).

• If \( f_1(x^*) \) is diagonalizable, it is then very easy to get an approximate solution.
  • Write \( f_1(x^*) = VDV^{-1} \), and define \( y = V^{-1}(x - x^*) \). Then \( \dot{y} = V^{-1}\dot{x} = V^{-1}f(x) \approx V^{-1}f_1(x^*)(x - x^*) = V^{-1}VDV^{-1}(x - x^*) = Dy \).
  • Hence, \( y_i(t) = y_{i,0}e^{Diit} \), from which it is easy to work out \( x = Vy + x^* \).

• However, unlike with discrete time models, it’s also quite easy to solve the system fully nonlinearly.
  • A crude algorithm (the Euler method) discretises time and treats the model as \( \Delta x_t = hf(x_{t-1}) \), where \( h \) is the time step. This is based on a first order approximation to the derivative.
  • More accurate approximations to the derivative deliver more accurate measures.
  • MATLAB contains many different ODE solvers. ode45 is a good starting point.
Hamilton-Jacobi-Bellman (HJB) equations: Finite horizon case

• Just as in discrete time, we can also tackle optimal control problems via a Bellman equation approach.

• Suppose:

\[
V(t, x(t)) = \max_c \left[ \int_t^T Y(\tau, x(\tau), c(\tau)) d\tau + \Psi(x(T)) \right]
\]

  • subject to the constraint that \( \dot{x}(t) = \Phi(t, x(t), c(t)) \).

• Then, for small (infinitesimal) \( dt \):

\[
V(t, x) = \max_c [Y(t, x, c) \ dt + V(t + dt, x + \Phi(t, x, c) \ dt)]
\]

• I.e.:

\[
0 = \max_c \left[ Y(t, x, c) + \frac{V(t + dt, x + \Phi(t, x, c) \ dt) - V(t, x)}{dt} \right]
\]

• Hence (or at least by this intuition), the HJB partial differential equation (PDE) is:

\[
-V_1(t, x) = \max_c [Y(t, x, c) + V_2(t, x)\Phi(t, x, c)],
\]

  • which must be solved subject to the terminal condition \( V(T, x) = \Psi(x(T)) \).
  • \( c \) will satisfy the standard FOC: \( Y_3(t, x, c) + V_2(t, x)\Phi_3(t, x, c) = 0 \).

• Whereas the previous method, based on Euler-Lagrange equations, gave necessary conditions for optimality, the HJB equation gives necessary and sufficient conditions, when solved globally.
HJB equations: Infinite horizon case

• Suppose:
\[ V(t, x(t)) = \max_c \left[ \int_t^\infty e^{-\rho \tau} u(x(\tau), c(\tau)) \, d\tau \right] \]

  • subject to the constraint that \( \dot{x}(t) = f(x(t), c(t)) \).

• By the same steps as before, this gives an HJB equation of the form:
\[ -V_1(t, x) = \max_c [e^{-\rho t} u(x, c) + V_2(t, x)f(x, c)] \]

• We then make the informed guess that \( V(t, x) = e^{-\rho t} V(x) \).
  • This implies that \( V_1(t, x) = -\rho V(t, x) \), and that \( V_2(t, x) = e^{-\rho t} V_1(x) \).

• Hence:
\[ \rho V(x) = \max_c [u(x, c) + V_1(x)f(x, c)] , \]

  • where \( V(t, x) = e^{-\rho t} V(x) \).
  • \( c \) will satisfy the standard FOC: \( u_2(x, c) + V_1(x)f_2(x, c) = 0 \).

• This is connected to our previous current-value Hamiltonian method through the substitution \( \lambda = V_1(x) \), since \( H_{c,2}(x, c, \lambda) = u_2(x, c) + \lambda f_2(x, c) \).

• Thus the HJB equation is just \( \rho V(x) = \max_c H_c(x, c, V_1(x)) \).
Solving HJB equations

• Global numerical techniques proceed (as in discrete time) by approximating the value function over a grid.

• As before, it is also possible to solve locally by treating the HJB equation and the FOC for $c$ as a system of ODEs.

• For some very simple models, analytic solutions may be derived via a “guess and verify” approach.

• For example, consider again the asset eating problem with $u(x, c) = \log c$, and $f(x, c) = rx - c$.
  • Then the HJB equation is: $\rho V(x) = \max_c [\log c + V_1(x)(rx - c)]$.
  • Let’s make the (informed) guess, $V(x) = a + b \log x$.
  • Then the FOC for $c$ gives $\frac{1}{c} = \frac{b}{x}$, so $c = \frac{x}{b}$.
  • Substituting in, we have $\rho (a + b \log x) = \log x - \log b + \frac{b}{x} (rx - \frac{x}{b})$, so clearly $b = \frac{1}{\rho}$ (so $c = \rho x$ as before) and: $a = \frac{1}{\rho} \left[ \log \rho + \frac{r}{\rho} - 1 \right]$. 
Multivariate Ito’s lemma

• Suppose:
  \[ dX_t = \mu_t \, dt + \sigma_t \, dW_t \]
  
  where \( X_t, \mu_t \in \mathbb{R}^n, \sigma_t \in \mathbb{R}^{n \times n} \) and \( W_t \) is an \( n \) dimensional vector of independent Brownian motions.

• Then, if \( f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \):

  \[
  df(t, X_t) = \left( f_1(t, X_t) + f_2(t, X_t)\mu_t + \frac{1}{2} \text{tr}(\sigma_t' f_2' \sigma_t) \right) dt + f_2(t, X_t) \sigma_t \, dW_t,
  \]

  where \( f_2' \) is the Hessian of \( f \) with respect to its second argument.
Stochastic HJB equations

• We just show the infinite horizon case here. Suppose:

\[ V(t, x(t)) = \max_c \mathbb{E}_t \left[ \int_t^\infty e^{-\rho \tau} u(x(\tau), c(\tau)) d\tau \right] \]

• subject to the constraint that \( dx(t) = f(x(t), c(t)) dt + \sigma(x(t), c(t)) dW(t) \).

• Then the (non-stochastic!) HJB equation is:

\[ \rho V(x) = \max_c \left[ u(x, c) + V_1(x) f(x, c) + \frac{1}{2} \text{tr} (\sigma(x, c)' V_{1'1}(x) \sigma(x, c)) \right], \]

• where \( V(t, x) = e^{-\rho t} V(x) \), where \( V_{1'1}(x) \) is the Hessian of \( V \).
• You will recognise the final term from Ito’s lemma.
• \( c \) will satisfy the standard FOC: \( u_2(x, c) + V_1(x) f_2(x, c) + V_{1'1}(x) \sigma(x, c)' \sigma_2(x, c) = 0 \).

• The remarkable thing is that the stochastic system has been converted to a non-stochastic set of ODEs.
Applications in macroeconomics of the stochastic HJB

- Olaf Posch has pioneered the recent application of the stochastic HJB in macroeconomics, building on earlier work by Merton.
- Richer models may be solved analytically in continuous time.

- Non-linearities are far easier to handle in continuous time.
  - See e.g. Fernandez-Villaverde, Posch and Rubio-Ramirez (2012) who solve an NK model with the ZLB in continuous time, getting analytic results for a special case, and accurate numerical results more generally.
  - Or Posch (2010) which derives analytic expression for general equilibrium risk premia, based on an extension of the Merton (1975) model.

- In the next few slides we’ll analyse the original Merton (1975) model.
Stochastic Ramsey model (Merton 1975) (1/3)

• We modify the set-up of the previous Ramsey-Cass-Koopmans model so that the evolution of labour and productivity are stochastic. In particular:

\[ d \log A(t) = \left( g - \frac{1}{2} \sigma_A^2 \right) dt + \sigma_A dW_A(t), \]
\[ d \log L(t) = \left( n - \frac{1}{2} \sigma_L^2 \right) dt + \sigma_L dW_L(t). \]

• So from Ito’s lemma:

\[ dA(t) = gA(t) dt + \sigma A(t) dW_A(t), \]
\[ dL(t) = nL(t) dt + \sigma L(t) dW_L(t). \]

• Recall:

\[ \dot{K} = K^\alpha (AL)^{1-\alpha} - C - \delta K \]

• Also, from Ito’s lemma:

\[ dk = d(K e^{-\log A - \log L}) \]
\[ = \left( \frac{\dot{K}}{AL} - \left( g - \frac{1}{2} \sigma_A^2 \right) \frac{K}{AL} - \left( n - \frac{1}{2} \sigma_L^2 \right) \frac{K}{AL} + \frac{1}{2} \left( \sigma_A^2 + \sigma_L^2 \right) \frac{K}{AL} \right) dt - \frac{K}{AL} \sigma_A dW_A(t) - \frac{K}{AL} \sigma_L dW_L(t) \]
\[ = (k^\alpha - c - (\delta + g + n + \sigma_A^2 + \sigma_L^2)k) dt - \sigma_A k dW_A(t) - \sigma_L k dW_L(t). \]
Stochastic Ramsey model (2/2)

• Setting the CRRA to $\alpha$ (special case!), the HJB equation is:

$$\rho V(k) = \max_c \left[ \frac{(cA(t))^{1-\alpha} - 1}{1 - \alpha} + V_1(k)(k^{\alpha} - c - (\delta + g + n + \sigma_A^2 + \sigma_L^2)k) \right]$$
Stochastic Ramsey model (2/2)

- Substituting this in gives:

\[
\rho a_0 + \rho a_1 A(t)^{1-\alpha} + \rho b A(t)^{1-\alpha} k^{1-\alpha}
\]

\[
= \left( (1 - \alpha)^{-\frac{1}{\alpha}} \frac{1}{b} \frac{1}{ka(t)} \right)^{1-\alpha} - 1 + (1 - \alpha) b A(t)^{1-\alpha} k^{-\alpha} \left( k^{\alpha} - \left( (1 - \alpha)^{-\frac{1}{\alpha}} \frac{1}{b} \frac{1}{a} + \delta + g + n + \sigma_A^2 + \sigma_L^2 \right) k \right)
\]

\[- \frac{1}{2} \alpha (1 - \alpha) b A(t)^{1-\alpha} k^{1-\alpha} (\sigma_A^2 + \sigma_L^2).
\]

- Matching terms in \( A(t)^{1-\alpha} k^{1-\alpha} \) gives:

\[
\rho = (1 - \alpha)^{-\frac{1}{\alpha}} \frac{1}{b} \frac{1}{a} - (1 - \alpha) (1 - \alpha)^{-\frac{1}{\alpha}} \frac{1}{b} \frac{1}{a} - (1 - \alpha) (\delta + g + n + \sigma_A^2 + \sigma_L^2) - \frac{1}{2} \alpha (1 - \alpha) (\sigma_A^2 + \sigma_L^2).
\]

- I.e.: \( b = \frac{\alpha}{1-\alpha} \left[ \rho + (1 - \alpha)(\delta + g + n + \sigma_A^2 + \sigma_L^2) + \frac{1}{2} \alpha (1 - \alpha) (\sigma_A^2 + \sigma_L^2) \right]^{-\alpha}. \)

- Exercise: find \( a_0, a_1 \).