Online appendices to “Reconciling near trend-stationary growth with medium-frequency cycles” and “Data consistent modelling of medium-frequency cycles and their origins”.

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Abstract: *This paper presents the online appendices to Holden (2013a) and Holden (2013b). We discuss the derivation of the first order and free-entry conditions, the steady state level of relative productivity of non-protected industries, and the nature of the inventor-firm bargaining procedure. We go on to present the full equations of both models considered, details of the data used for estimation, and the results of this estimation procedure.*

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1. The free-entry and first order conditions

When deciding how much research and appropriation to perform, firms realise that if they perform a non-equilibrium amount then in the next period they will have an incentive to set a different mark-up to the other firms in their industry. (The clearest example of this is when we have perfect competition, in which case the most productive firm would want to price just below the second most productive firms' marginal cost.) It may be seen that in non-symmetric equilibrium the optimal price satisfies:

$$P_t(i,j) = \frac{W_t}{A_t(i,j)} \left[1 + \frac{\eta\lambda}{1 - (1-\eta)\frac{1}{J_{t-1}(i)}\left(\frac{P_t(i,j)}{P_t(i)}\right)^{-\frac{1}{\eta\lambda}}} \right].$$

Since we are looking for a symmetric equilibrium, it is sufficient to approximate this locally around $P_t(i) = P_t(i,j)$ in order to calculate firms' research and appropriation incentives. Taking a log-linear approximation of $\log P_t(i,j)$ in $\frac{P_t(i,j)}{P_t(i)}$ gives us that:

$$P_t(i,j) \approx \frac{W_t}{A_t(i,j)} \left(1 + \mu_{t-1}(i)\right) \left(\frac{P_t(i,j)}{P_t(i)}\right)^{-\omega_{t-1}(i)}$$

where $\omega_t(i) := \frac{J_t(i)(1-\eta)}{\left(J_t(i)-(1-\eta)\right)^2 \left(1+\mu_t(i)\right)}$ captures the strength of these incentives to deviate from setting the same mark-up as all other firms in their industry. Therefore $P_t(i) \approx \frac{W_t}{A_t(i)}\left(1 + \mu_{t-1}(i)\right)$ and $P_t(i,j) \approx \frac{W_t}{A_t(i,j)}\left(1 + \mu_{t-1}(i)\right)\left(\frac{A_t(i,j)}{A_t(i)}\right)^{\frac{\omega_{t-1}(i)}{1+\omega_{t-1}(i)}}$ where:

$$A_t(i) := \left[\frac{1}{J_{t-1}(i)} \sum_{j=1}^{J_{t-1}(i)} A_t(i,j)^{\frac{1}{\eta\lambda(1+\omega_{t-1}(i))}} \right]^{\eta\lambda(1+\omega_{t-1}(i))}.$$

Therefore, up to a first order approximation around the symmetric solution, profits are given by:

$$\beta \frac{1}{I_t J_t(i)} \left(\frac{1+\mu_t}{1+\mu_t(i)}\right)^{\frac{1}{\lambda}} \mathbb{E}_t \Xi_{t+1} Y_{t+1} \left[\left(\frac{A_{t+1}(i,j)}{A_{t+1}(i)}\right)^{\frac{\omega_t(i)}{1+\omega_t(i)}} - \frac{1}{1+\mu_t(i)} \right] \left(\frac{A_{t+1}(i,j)}{A_{t+1}(i)}\right)^{\frac{1-\eta\lambda\omega_t(i)}{\eta\lambda(1+\omega_t(i))}} \left(\frac{A_{t+1}(i)}{A_{t+1}}\right)^{\frac{1}{\lambda}} - \left[L_t^{\mathrm{R}}(i,j) + L_t^{\mathrm{A}}(i,j) + L_t^{\mathcal{R}}(i) + L^{\mathrm{F}}\right] W_t.$$

Note that if $J_t(i) > \frac{2\sqrt{2}(3-\sqrt{2})}{1+2\sqrt{2}} \approx 1.17$, then $1 - \eta\lambda\omega_t(i) > 0$ (by tedious algebra), so providing there are at least two firms in the industry, this expression is guaranteed to be increasing and concave in $A_{t+1}(i,j)$.

Let $\boldsymbol{m}_t^{\mathrm{R}}(i,j)W_t$ be the Lagrange multiplier on research's positivity constraint and $\boldsymbol{m}_t^{\mathrm{A}}(i,j)W_t$ be the Lagrange multiplier on appropriation's positivity constraint. Then in a symmetric equilibrium the two first order conditions and the free entry condition (respectively) mean:

$$\beta\frac{1}{I_t J_t(i)}\frac{\mu_t(i)}{1+\mu_t(i)}\left(\frac{1+\mu_t}{1+\mu_t(i)}\right)^{\frac{1}{\lambda}}\mathbb{E}_t\Xi_{t+1}Y_{t+1}\left(\frac{A_{t+1}(i)}{A_{t+1}}\right)^{\frac{1}{\lambda}}\frac{\boldsymbol{d}_t(i)}{\mu_t(i)}\frac{Z_{t+1}A_t^{**}(i)^{-\zeta^{\mathrm{R}}}\Psi}{1+\gamma Z_{t+1}A_t^{**}(i)^{-\zeta^{\mathrm{R}}}\Psi L_t^{\mathrm{R}}(i)}$$
$$= W_t\big(1-\boldsymbol{m}_t^{\mathrm{R}}(i)\big)$$
$$\beta\frac{1}{I_t J_t(i)}\frac{\mu_t(i)}{1+\mu_t(i)}\left(\frac{1+\mu_t}{1+\mu_t(i)}\right)^{\frac{1}{\lambda}}\mathbb{E}_t\Xi_{t+1}Y_{t+1}\left(\frac{A_{t+1}(i)}{A_{t+1}}\right)^{\frac{1}{\lambda}}\frac{\boldsymbol{d}_t(i)}{\mu_t(i)}\frac{1+(\gamma-\zeta^{\mathrm{R}})Z_{t+1}A_t^{**}(i)^{-\zeta^{\mathrm{R}}}\Psi L_t^{\mathrm{R}}(i)}{1+\gamma Z_{t+1}A_t^{**}(i)^{-\zeta^{\mathrm{R}}}\Psi L_t^{\mathrm{R}}(i)}$$
$$\cdot\frac{1}{\tau}\frac{A_t(i)^{-\zeta^{\mathrm{A}}}\Upsilon(A_t^{*\tau}-A_t(i)^{\tau})}{A_t^{**}(i)^{\tau}\left(1+A_t(i)^{-\zeta^{\mathrm{A}}}\Upsilon L_t^{\mathrm{A}}(i)\right)^2} = W_t\left(1-\boldsymbol{m}_t^{\mathrm{A}}(i)\right)$$
$$\beta\frac{1}{I_t J_t(i)}\frac{\mu_t(i)}{1+\mu_t(i)}\left(\frac{1+\mu_t}{1+\mu_t(i)}\right)^{\frac{1}{\lambda}}\mathbb{E}_t\Xi_{t+1}Y_{t+1}\left(\frac{A_{t+1}(i)}{A_{t+1}}\right)^{\frac{1}{\lambda}}$$
$$= [L_t^{\mathrm{R}}(i,j)+L_t^{\mathrm{A}}(i,j)+L_t^{\mathcal{R}}(i)+L_t^{\mathrm{F}}]W_t$$

where:

$$\boldsymbol{d}_t(i) := 1-\frac{\omega_t(i)}{1+\omega_t(i)}\frac{\big(\lambda-\mu_t(i)\big)(\mu_t(i)-\eta\lambda)}{\lambda(1-\eta)\mu_t(i)} < 1$$

and where we have dropped j indices on variables which are the same across the industry.

We also have that:

$$\frac{\big(\lambda-\mu_t(i)\big)(\mu_t(i)-\eta\lambda)}{\lambda(1-\eta)\mu_t(i)} \le \frac{\lambda(1-\sqrt{\eta})(\sqrt{\eta}-\eta)}{\sqrt{\eta}} < \lambda$$

so providing $\lambda < 1$, $\boldsymbol{d}_t(i) > 0$.

That the solution for research when $Z_{t+1} \equiv 1$ is given by equation (1.2) from Holden (2013a) is a trivial consequence of the complementary slackness condition and the facts that $\frac{1}{\mu_t(i)} < \gamma$ and $\boldsymbol{d}_t(i) < 1$. Deriving (1.3) from Holden (2013a) is less trivial though.

Begin by defining $\boldsymbol{k}_t(i) := \frac{1+(\gamma-\zeta^{\mathrm{R}})\mathcal{L}_t^{\mathrm{R}}(i)}{1+\gamma\mathcal{L}_t^{\mathrm{R}}(i)}$, and note that since we are assuming $\gamma > \zeta^{\mathrm{R}} \ge 0$, we have that $0 < \boldsymbol{k}_t(i) \le 1$. Also define:

$$\boldsymbol{n}_t(i) := \frac{\boldsymbol{d}_t(i)\boldsymbol{k}_t(i)}{\tau\mu_t(i)}A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon\left[\left(\frac{A_t^*}{A_t^*(i)}\right)^{\tau}-1\right][L_t^{\mathrm{R}}(i)+L_t^{\mathcal{R}}(i)+L_t^{\mathrm{F}}] \ge 0,$$

which is not a function of $L_t^{\mathrm{A}}(i)$, given $L_t^{\mathrm{R}}(i)$.

We can then combine the appropriation first order condition with the free entry condition to obtain:

$$\frac{1}{\left(1+\mathcal{L}_t^{\mathrm{A}}(i)\right)^2}\left(\frac{A_t^*(i)}{A_t^{**}(i)}\right)^\tau\left[\frac{\boldsymbol{d}_t(i)\boldsymbol{k}_t(i)}{\tau\mu_t(i)}\left[\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau-1\right]\mathcal{L}_t^{\mathrm{A}}(i)+\boldsymbol{n}_t(i)\right]$$
$$=1-\boldsymbol{m}_t^{\mathrm{A}}(i).$$

Since the left hand side is weakly positive, from the dual feasibility condition we know $\boldsymbol{m}_t^{\mathrm{A}}(i)\in[0,1]$. Now when $L_t^{\mathrm{A}}(i)=0$, this becomes:

$$\boldsymbol{n}_t(i)=1-\boldsymbol{m}_t^{\mathrm{A}}(i),$$

since in this case $A_t^*(i)=A_t^{**}(i)$. Therefore when $L_t^{\mathrm{A}}(i)=0$, $\boldsymbol{n}_t(i)\le 1$.

We now prove the converse. Suppose then for a contradiction that $L_t^{\mathrm{A}}(i)>0$, but $\boldsymbol{n}_t(i)\le 1$. By complementary slackness, we must have $\boldsymbol{m}_t^{\mathrm{A}}(i)=0$, hence:

$$\begin{aligned}1\ge\boldsymbol{n}_t(i)&=\left(1+\mathcal{L}_t^{\mathrm{A}}(i)\right)^2\left(\frac{A_t^{**}(i)}{A_t^*(i)}\right)^\tau-\frac{\boldsymbol{d}_t(i)\boldsymbol{k}_t(i)}{\tau\mu_t(i)}\left[\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau-1\right]\mathcal{L}_t^{\mathrm{A}}(i)\\
&\ge\left(1+\mathcal{L}_t^{\mathrm{A}}(i)\right)^2\left(\frac{A_t^{**}(i)}{A_t^*(i)}\right)^\tau-\left[\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau-1\right]\mathcal{L}_t^{\mathrm{A}}(i)\\
&=\left(1+\mathcal{L}_t^{\mathrm{A}}(i)\right)\left[\left(1+\mathcal{L}_t^{\mathrm{A}}(i)\right)+\mathcal{L}_t^{\mathrm{A}}(i)\left[\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau-1\right]\right]\\
&\qquad-\left[\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau-1\right]\mathcal{L}_t^{\mathrm{A}}(i),\end{aligned}$$

where we have used the facts that $\boldsymbol{d}_t(i)\boldsymbol{k}_t(i)\le 1$ and $\frac{1}{\mu_t(i)}<\tau$ to derive the second inequality.

Expanding the brackets then gives that:

$$1\ge 1+2\mathcal{L}_t^{\mathrm{A}}(i)+\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau\mathcal{L}_t^{\mathrm{A}}(i)^2,$$

i.e. that $0\ge 2+\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau\mathcal{L}_t^{\mathrm{A}}(i)$ which is a contradiction as $\left(\frac{A_t^*}{A_t^*(i)}\right)^\tau\mathcal{L}_t^{\mathrm{A}}(i)\ge 0$.

We have proven then that providing $\frac{1}{\mu_t(i)}<\tau$, $L_t^{\mathrm{A}}(i)=0$ if and only if $\boldsymbol{n}_t(i)\le 1$. It just remains for us to solve for $L_t^{\mathrm{A}}(i)$ when it is strictly positive. From the above, we have that, in this case:

$$\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau\left[\boldsymbol{n}_t(i)-1\right]$$
$$=2\left[1-\frac{1}{2}\left[1+\frac{\boldsymbol{d}_t(i)\boldsymbol{k}_t(i)}{\tau\mu_t(i)}\right]\left[1-\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau\right]\right]\mathcal{L}_t^{\mathrm{A}}(i)+\mathcal{L}_t^{\mathrm{A}}(i)^2.$$

Hence:

$$\begin{aligned}&\mathcal{L}_t^{\mathrm{A}}(i)\\&=-\left[1-\frac{1}{2}\left[1+\frac{d_t(i)k_t(i)}{\tau\mu_t(i)}\right]\left[1-\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau\right]\right]\\&+\sqrt{\left[1-\frac{1}{2}\left[1+\frac{d_t(i)k_t(i)}{\tau\mu_t(i)}\right]\left[1-\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau\right]\right]^2+\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau[n_t(i)-1]},\end{aligned}$$

since the lower solution is guaranteed to be negative as $n_t(i)>1$ when $L_t^{\mathrm{A}}(i)>0$.

2. The steady state for non-patent-protected industries

In an industry i which is not patent-protected and in which appropriation, but no research, is performed, from (1.1) and (1.3) of Holden (2013a):

$$f_t(i)+\sqrt{f_t(i)^2+g_t(i)}=\mathcal{L}_t^{\mathrm{A}}(i)=\left[1-\frac{\left(\frac{A_{t+1}^*(i)}{A_t^*(i)}\right)^\tau-1}{1-\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau}\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau\right]^{-1}-1.$$

If we treat $\mathfrak{p}_1:=\tau\frac{\mu_t(i)}{d_t(i)}-1\approx 0$, $\mathfrak{p}_2:=A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon_t L_t^{\mathrm{F}}\approx 0$ and $\mathfrak{p}_3:=\left(\frac{A_{t+1}^*(i)}{A_t^*(i)}\right)^\tau-1\approx 0$ as fixed, this leaves us with a cubic in $\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau$, for which only one solution will be feasible (i.e. strictly less than 1). Taking a second order Taylor approximation of this solution in $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$, reveals (after some messy computation), that:

$$\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau\approx\mathfrak{p}_2\big(1-(\mathfrak{p}_1+\mathfrak{p}_2)\big)=A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon_t L_t^{\mathrm{F}}\left(2-\tau\frac{\mu_t(i)}{d_t(i)}-A_t^*(i)^{-\zeta^{\mathrm{A}}}\Upsilon_t L_t^{\mathrm{F}}\right)$$

(The effect of $\mathfrak{p}_3$ on $\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau$ is third order and hence it does not appear in this expression.)

From this approximate solution for $\left(\frac{A_t^*(i)}{A_t^*}\right)^\tau$ then, we have that the relative productivity of a non-protected industry is decreasing in its mark-up. Furthermore, from dropping to a first order approximation, we have that $A_t^*(i)^{1+\frac{\zeta^{\mathrm{A}}}{\tau}}\approx A_t^*(\Upsilon_t L_t^{\mathrm{F}})^{\frac{1}{\tau}}$, so asymptotically non-protected industries are growing at $\left[1+\frac{\zeta^{\mathrm{A}}}{\tau}\right]^{-1}$ times the growth rate of the frontier.

3. The inventor-firm bargaining process

We model the entire process of setting and paying rents as follows:

1) Firms enter, paying the fixed cost.
2) Firms who have entered conduct appropriation, then research.
3) The "idea shock" for next period's production, Z_{t+1}, is realised and firms and patent holders learn its level.
4) Finally, firms arrive at the patent-holder to conduct bargaining, with these arrivals taking place sequentially but in a random order. (For example, all firms phone the patent-holder sometime in the week before production is to begin.) In this bargaining we suppose that the patent-holder has greater bargaining power, since they have a longer outlook[2] and since they lose nothing if bargaining collapses[3]. We also suppose that neither patent-holders nor firms are able to observe or verify either how many (other) firms paid the fixed cost, or what research and appropriation levels they chose. This is plausible because until production begins it is relatively easy to keep such things hidden (for example, by purchasing the licence under a spin-off company), and because it is hard to ascertain ahead of production exactly what product a firm will be producing. We assume bargaining takes an alternating offer form, (Rubinstein 1982) but that it happens arbitrarily quickly (i.e. in the no discounting limit).
5) Firms pay the agreed rents if bargaining was successful. Since this cost is expended before production, we continue to suppose firms have to borrow in the period before production in order to cover it. Firms will treat it as a fixed cost, sunk upon entry, since our unobservability assumptions mean bargaining's outcome will not be a function of research and appropriation levels.
6) The next period starts, other aggregate shocks are realised and production takes place.

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[2] Consider what happens as the time gap between offers increases. When this gap is large enough only one offer would be made per-period, meaning the patent-holder would make a take-it-or-leave-it offer giving (almost) nothing to the firm, which the firm would then accept.

[3] The firm owner may, for example, face restrictions from starting businesses in future if as a result of the bargaining collapse they are unable to repay their creditors.

7) The patent-holder brings court cases against any firms who produced but decided not to pay the rent. For simplicity, we assume the court always orders the violating firm to pay damages to the patent-holder, which are given as follows:
   a) When the courts believe rents were not reasonable (i.e. $L_t^{\mathcal{R}}(i) > L_t^{\mathcal{R}*}(i)$, where $L_t^{\mathcal{R}*}(i)W_t$ is the level courts determine to be "reasonable royalties"), they set damages greater than $L_t^{\mathcal{R}*}(i)W_t$, as "*the infringer would have nothing to lose, and everything to gain if he could count on paying only the normal, routine royalty non-infringers might have paid*"[4]. We assume excess damages over $L_t^{\mathcal{R}*}(i)W_t$ are less than the patent-holder's legal costs however.
   b) When the courts consider the charged rent to have been reasonable (i.e. $L_t^{\mathcal{R}}(i) \le L_t^{\mathcal{R}*}(i)$) the courts award punitive damages of more than $\max\left\{ L_t^{\mathcal{R}*}(i)W_t, \left(\frac{1}{1-\wp}\right)L_t^{\mathcal{R}}(i)W_t, \right\}$, where $\wp$ is the bargaining power of the firm, in the sense of the generalized Nash bargaining solution.[5]

Under this specification:

$$L_t^{\mathcal{R}}(i) = \min\{L_t^{\mathcal{R}*}(i), (1-\wp)[L_t^{\mathrm{R}}(i) + L_t^{\mathrm{A}}(i) + L_t^{\mathcal{R}}(i) + L^{\mathrm{F}}]\}$$

since entry is fixed when bargaining takes place, since patent-holders know that bargaining to a rent level any higher than $L_t^{\mathcal{R}*}(i)W_t$ will just result in them having to pay legal costs,[6] and since $[L_t^{\mathrm{R}}(i) + L_t^{\mathrm{A}}(i) + L_t^{\mathcal{R}}(i) + L^{\mathrm{F}}]W_t$ is equal to the production period profits of each firm in industry i, by the free entry condition.[7] Therefore, in equilibrium:

$$L_t^{\mathcal{R}}(i) = \min\left\{L_t^{\mathcal{R}*}(i), L_t^{\mathcal{R}\dagger}(i)\right\}, \tag{3.1}$$

[4] Panduit Corp. v. Stahlin Brothers Fibre Works, Inc., 575 F.2d 1152, 1158 (6th Circuit 1978), cited in Pincus (1991).

[5] The level $\left(\frac{1}{1-\wp}\right)L_t^{\mathcal{R}}(i)W_t$ is chosen to ensure that, with equilibrium rents, firms prefer not to produce at all rather than to produce without paying rents.

[6] The disagreement point is zero since it is guaranteed that $L_t^{\mathfrak{R}}(i) \le L_t^{\mathfrak{R}*}(i)$ and so punitive damages would be awarded were the firm to produce without paying rents, which, by construction, leaves them worse off than not producing.

[7] A similar expression can also be derived if we assume instead that courts guarantee infringers a fraction $\wp$ of production profits, or if we assume courts always award punitive damages but firms are able to hide a fraction $\wp$ of their production profits.

where $L_t^{\mathcal{R}\dagger}(i)$ is a solution to equations (1.2), (1.3) and (1.4) from Holden (2013a), (i.e. $L_t^{\mathcal{R}}(i) = \frac{1-\wp}{\wp}[L_t^{\mathrm{R}}(i) + L_t^{\mathrm{A}}(i) + L^{\mathrm{F}}]$) if one exists, or $+\infty$ otherwise. Because damages are always greater than $L_t^{\mathcal{R}*}(i)W_t$, these rents will be sufficiently low to ensure firms are always prepared to licence the patent at the bargained price in equilibrium.

Now suppose we are out of equilibrium and fewer firms than expected have entered. Since neither the patent-holder nor firms can observe how many firms have entered, and since firms arrive at the patent-holder sequentially, both sides will continue to believe that the equilibrium number of firms has entered and so rents will not adjust. On the other hand, suppose that (out of equilibrium) too many firms enter. When the first unexpected firm arrives at the patent-holder to negotiate, the patent-holder will indeed realise that too many firms have entered. However, since the firm they are bargaining with has no way of knowing this,[8] the patent-holder can bargain for the same rents as in equilibrium. Therefore, even out of equilibrium:

$$L_t^{\mathcal{R}}(i) = \min\left\{L_t^{\mathcal{R}*}(i), L_t^{\mathcal{R}\dagger}(i)\right\}$$

where we stress $L_t^{\mathcal{R}\dagger}(i)$ is not a function of the decisions any firm happened to take. This ensures that any solution of equation (3.1) with equations (1.2) and (1.3) from Holden (2013a), will also be an equilibrium, even allowing for the additional condition that the derivative of firm profits with respect to the number of firms must be negative at an optimum.

We now just have to pin down "reasonable royalties", $L_t^{\mathcal{R}*}(i)W_t$. *Georgia-Pacific,* 318 F. Supp. at 1120 (S.D.N.Y. 1970), *modified on other grounds,* 446 F.2d 295 (2d Cir.), *cert. denied,* 404 U.S. 870 (1971), cited in Pincus (1991), defines a reasonable royalty as "*the amount that a licensor (such as the patentee) and a licensee (such as the infringer) would have agreed upon (at the time the infringement began) if both had been reasonably and voluntarily trying to reach an agreement; that is, the amount which a prudent licensee—who desired, as a*

[8] Either they are a firm that thinks the equilibrium number of firms has entered, or they are a firm that thinks more than the equilibrium number of firms has entered, but does not know whether the patent-holder has yet realised this.

business proposition, to obtain the licence to manufacture and sell a particular article embodying the patented invention—would have been willing to pay as a royalty and yet be able to make a reasonable profit and which amount would have been acceptable by a prudent patentee who was willing to grant a licence."

Certainly it must be the case that $L_t^{\mathcal{R}*}(i) \leq L_t^{\bar{\mathcal{R}}}(i)$, where $L_t^{\bar{\mathcal{R}}}(i)$ is the level of rents at which $J_t(i) = 1$, since rents so high that no one is prepared to pay them must fall foul of the courts' desire to ensure licensees can make a profit. [9] However, since when $J_t(i) = 1$ the sole entering firm (almost) may as well be the patent-holder themselves, where possible the courts will set $L_t^{\mathcal{R}*}(i)$ sufficiently low to ensure that $J_t(i) > 1$ in equilibrium, again following the idea that licensees ought to be able to make a profit. When there is a $J_t(i) > 1$ solution to equations (1.2), (1.3) and (1.4) from Holden (2013a) already (i.e. $L_t^{\mathcal{R}\dagger}(i) < \infty$), the courts will just set $L_t^{\mathcal{R}*}(i)$ at the rent level that would obtain in that solution, thus preventing the possibility of $J_t(i) = 1$ being an equilibrium. It may be shown that for sufficiently large t such a solution is guaranteed to exist, so in this case $L_t^{\mathcal{R}*}(i) = L_t^{\mathcal{R}\dagger}(i) = L_t^{\mathcal{R}}(i)$.[10]

4. The de-trended model

Below we give the equations of the stationary model to which the model described in section 2.1 of Holden (2013a) converges as $t \to \infty$.

4.1. Households

- **Stochastic discount factor:** $\Xi_t = \frac{\hat{C}_{t-1}}{\hat{C}_t G_{A,t}}$, where $\hat{C}_t := \frac{C_t}{N_t A_t}$ is consumption per person in labour supply units and $G_{V,t}$ is the exponent of the growth rate of the variable V_t at t.

[9] "*…the very definition of a reasonable royalty assumes that, after payment, the infringer will be left with a profit.*" Georgia-Pacific Corp. v. U.S. Plywood-Champion Papers Corp., 446 F.2d 295, 299 & n.1 (2d Cir.), *cert. denied,* 404 U.S. 870 (1971), cited in Pincus (1991).

[10] There may still be multiple solutions for rents (as (1.2), (1.3) and (1.4) from Holden (2013a) might have multiple solutions), but of these only the one with minimal entry is really plausible, since this is both weakly Pareto dominant (firms always make zero profits and it may be shown that the patent-holder prefers minimal entry) and less risky for entering firms (if entering firms are unsure if the patent-holder will play the high rent or the low rent equilibrium, they are always better off assuming the high rent one since if that assumption is wrong they make strict profits, whereas had they assumed low rents but rents were in fact high they would make a strict loss).

- **Labour supply:** $\hat{L}_t^{\mathrm{S}^\nu} = \frac{\widehat{W}_t}{\hat{C}_t}$, where $\hat{L}_t^{\mathrm{S}} := \frac{L_t^{\mathrm{S}}}{N_t}$ is labour supply per person and $\widehat{W}_t := \frac{W_t}{A_t}$ is the wage per effective unit of labour supply.
- **Euler equation:** $\beta R_t \mathbb{E}_t[\Xi_{t+1}] = 1$, where R_t is the real interest rate.

4.2. Aggregate relationships

- **Aggregate mark-up pricing:** $\widehat{W}_t = \frac{1}{1+\mu_{t-1}}$ where μ_{t-1} is the aggregate mark-up in period t.
- **Mark-up aggregation:** $\left(\frac{1}{1+\mu_t}\right)^{\frac{1}{\lambda}} = \left(\frac{1}{1+\mu_t^{\mathrm{P}}}\right)^{\frac{1}{\lambda}} \mathcal{s}_t + \left(\frac{1}{1+\eta\lambda}\right)^{\frac{1}{\lambda}} (1-\mathcal{s}_t)$, where $\mu_t^{\mathrm{P}} = \mu_t(I_t)$ is the mark-up in any protected industry at $t+1$, and $\mathcal{s}_t := (1-\mathcal{q}) \frac{\mathcal{s}_{t-1}}{G_{I,t}} + 1 - \frac{1}{G_{I,t}}$ is the proportion of industries that will produce a patent protected product in period $t+1$.
- **Productivity aggregation:** $\left(\frac{\hat{A}_t}{1+\mu_{t-1}}\right)^{\frac{1}{\lambda}} = \left(\frac{1}{1+\mu_{t-1}^{\mathrm{P}}}\right)^{\frac{1}{\lambda}} \mathcal{s}_{t-1} + \left(\frac{\hat{A}_t^{\mathrm{N}}}{1+\eta\lambda}\right)^{\frac{1}{\lambda}} (1 - \mathcal{s}_{t-1})$, where $\hat{A}_t := \frac{A_t}{A_t^*}$ is aggregate productivity relative to the frontier[11] and $\hat{A}_t^{\mathrm{N}} := \left[\left(\frac{1}{G_{A^*,t}}\right)^{\frac{1}{\lambda}} \left(\frac{\mathcal{q}}{1/\mathcal{s}_{t-2}-(1-\mathcal{q})}\right) + \left(\frac{\hat{A}_{t-1}^{\mathrm{N}}}{G_{A^*,t}}\right)^{\frac{1}{\lambda}} \left(1 - \frac{\mathcal{q}}{1/\mathcal{s}_{t-2}-(1-\mathcal{q})}\right)\right]^{\lambda}$ is the aggregate relative productivity of non-protected industries.

4.3. Firm decisions

- **Strategic in-industry pricing:** $\mu_t^{\mathrm{P}} = \lambda \frac{\eta \hat{J}_t^{\mathrm{P}}}{\hat{J}_t^{\mathrm{P}} - (1-\eta)}$, where $\hat{J}_t^{\mathrm{P}} := J_t(I_t)$ is the number of firms in a protected industry performing research at t.
- **Firm research decisions:** $\frac{d_t}{\wp \mu_t^{\mathrm{P}}} \mathbb{E}_t \Xi_{t+1} G_{Y,t+1} \hat{A}_{t+1}^{-\frac{1}{\lambda}} \frac{Z_{t+1}\hat{\mathcal{L}}_t^{\mathrm{R}}}{1+\gamma Z_{t+1}\hat{\mathcal{L}}_t^{\mathrm{R}}} = (1 - m_t^{\mathrm{R}}) \mathbb{E}_t \Xi_{t+1} G_{Y,t+1} \hat{A}_{t+1}^{-\frac{1}{\lambda}}$, where $\hat{\mathcal{L}}_t^{\mathrm{R}} := A_t^{*-\zeta} \Psi L_t^{\mathrm{R}}$ is the amount of effective research conducted by firms in protected industries d_t is the value of $d_t(i)$ in protected industries and Z_t is the aggregate research-return shock. (This equation means that $\hat{\mathcal{L}}_t^{\mathrm{R}} \approx \frac{\wp \mu_t^{\mathrm{P}}}{d_t - \wp\gamma\mu_t^{\mathrm{P}}}$.)
- **Research and appropriation payoff:** $G_{A^*,t} = \left(1 + \gamma Z_t \hat{\mathcal{L}}_{t-1}^{\mathrm{R}}\right)^{\frac{1}{\gamma}}$.
- **Free entry of firms:** $\beta \frac{1}{\hat{I}_t \hat{J}_t^{\mathrm{P}}} \frac{\mu_t^{\mathrm{P}}}{1+\mu_t^{\mathrm{P}}} \left(\frac{1+\mu_t}{1+\mu_t^{\mathrm{P}}}\right)^{\frac{1}{\lambda}} \mathbb{E}_t \Xi_{t+1} G_{Y,t+1} \hat{A}_{t+1}^{-\frac{1}{\lambda}} = \frac{1}{\wp} \hat{\mathcal{L}}_t^{\mathrm{R}} \frac{\widehat{W}_t}{\hat{Y}_t}$, where $\hat{I}_t := \frac{I_t}{N_t A_t^{*-\zeta}\Psi}$ is the measure of products relative to its trend,[12] and $\hat{Y}_t := \frac{Y_t}{N_t A_t}$ is output per person in labour supply units.

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[11] As a consequence, we have that $G_{A,t} = \frac{\hat{A}_t}{\hat{A}_{t-1}} G_{A^*,t}$.

[12] This means $G_{I,t} = G_{N,t} G_{A^*,t}^{-\zeta} \frac{\hat{I}_t}{\hat{I}_{t-1}}$.

4.4. Inventor decisions

- **Inventor profits:** are given recursively by: $\hat{\pi}_t = \frac{1-\mathscr{p}}{\mathscr{p}} \hat{\mathcal{L}}_t^{\mathrm{R}} \widehat{W}_t \hat{J}_t^{\mathrm{P}} + \beta(1-\mathscr{q})\mathbb{E}_t \Xi_{t+1} G_{A,t+1} G_{A^*,t+1}^{\zeta} \hat{\pi}_{t+1}$, where $\hat{\pi}_t := \frac{\pi_t}{A_t A_t^{*\zeta}}$.
- **Free entry of inventors:** Either $G_{I,t} \geq 1$ binds or $\Psi E^{\zeta} \mathcal{L}^{\mathrm{I}} \widehat{W}_t \geq \hat{\pi}_t$ does.

4.5. Market clearing

- **Labour market clearing:** $\hat{L}_t^{\mathrm{S}} = \Psi E^{\zeta} \mathcal{L}_t^{\mathrm{I}} \hat{I}_t \left(1 - \frac{1}{G_{I,t}}\right) + \hat{I}_t \mathscr{s}_t \hat{J}_t^{\mathrm{P}} \hat{\mathcal{L}}_t^{\mathrm{R}} + \hat{Y}_t \left[\left(\frac{1}{\hat{A}_t}\right)^{\frac{1}{\lambda_t}} \left(\frac{1+\mu_{t-1}}{1+\mu_{t-1}^{\mathrm{P}}}\right)^{\frac{1+\lambda}{\lambda}} \mathscr{s}_{t-1} + \left(\frac{\hat{A}_t^{\mathrm{N}}}{\hat{A}_t}\right)^{\frac{1}{\lambda_t}} \left(\frac{1+\mu_{t-1}}{1+\eta\lambda}\right)^{\frac{1+\lambda}{\lambda}} (1-\mathscr{s}_{t-1}) \right]$.
- **Goods market clearing:** $\hat{Y}_t = \hat{C}_t$.

5. The extended de-trended model

Define $\mathscr{a} := \frac{1}{(1-\alpha_{\mathrm{P}})(1-\iota_{\mathrm{P}})}$, $\mathscr{b} := (1-\alpha_{\mathrm{R}})(1-\iota_{\mathrm{R}})$, $\mathscr{c} := \left(\frac{1-\alpha_{\mathrm{R}}}{1-\alpha_{\mathrm{P}}} \alpha_{\mathrm{P}} \xi_{\mathrm{KP}} - \alpha_{\mathrm{R}} \xi_{\mathrm{KR}}\right)(1-\iota_{\mathrm{R}})$, $\mathscr{e} := \xi_{\mathrm{L}} + \frac{\alpha_{\mathrm{P}}}{1-\alpha_{\mathrm{P}}} \xi_{\mathrm{KP}}$ and make the normalisation $\Psi = E = 1$.

5.1. Households

- **Budget constraint Lagrange multiplier:** $\frac{1}{\hat{\tilde{C}}_t} = \widehat{\mathscr{m}}_t^{\mathrm{C}} + \beta \mathscr{h} \mathscr{h}^{\mathrm{INT}} \mathbb{E}_t \frac{N_{t+1}\Theta_{t+1}}{N_t \Theta_t} \frac{1}{G_{A,t+1}^{\mathscr{a}} \hat{\tilde{C}}_{t+1}}$, where $\frac{\widehat{\mathscr{m}}_t^{\mathrm{C}}}{A_t^{\mathscr{a}} A_t^{*\mathscr{e}} N_t}$ is the Lagrange multiplier on the budget constraint and $\hat{\tilde{C}}_t := \frac{\tilde{C}_t}{A_t^{\mathscr{a}} A_t^{*\mathscr{e}}} = \hat{C}_t - \mathscr{h} \frac{\hat{C}_{t-1}}{G_{A,t}^{\mathscr{a}} G_{A^*,t}^{\mathscr{e}}}$.
- **Stochastic discount factor:** $\Xi_t = \frac{\Theta_t \widehat{\mathscr{m}}_t^{\mathrm{C}}}{\Theta_{t-1} \widehat{\mathscr{m}}_{t-1}^{\mathrm{C}} G_{A,t}^{\mathscr{a}} G_{A^*,t}^{\mathscr{e}}}$.
- **Labour supply:** $(1+\lambda_{\mathrm{L}}) \widehat{\mathscr{w}}_{1,t} = \widehat{\widetilde{W}}_t^{1+\nu\frac{1+\lambda_{\mathrm{L}}}{\lambda_{\mathrm{L}}}} \widehat{\mathscr{w}}_{2,t}$, where $\widehat{\widetilde{W}}_t := \left[\frac{\widehat{W}_t^{-\frac{1}{\lambda_{\mathrm{L}}}} - \mathscr{v}\left(\frac{G_P}{G_{P,t}} \frac{G_W}{G_{A,t}^{\mathscr{a}} G_{A^*,t}^{\mathscr{e}}} \widehat{W}_{t-1}\right)^{-\frac{1}{\lambda_{\mathrm{L}}}}}{1-\mathscr{v}}\right]^{-\lambda_{\mathrm{L}}}$ ($\widehat{\widetilde{W}}_t A_t^{\mathscr{a}} A_t^{*\mathscr{e}}$ is the real wage set by a household that updates its wage at t), $\widehat{W}_t := \frac{W_t}{A_t^{\mathscr{a}} A_t^{*\mathscr{e}}}$, and where $\widehat{\mathscr{w}}_{1,t}$ and $\widehat{\mathscr{w}}_{2,t}$ are the sums of costs and benefits respectively from the wage setting first order conditions.[13]
- **Euler equation:** $\beta R_t \mathbb{E}_t [\Xi_{t+1}] = 1$.

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[13] $\widehat{\mathscr{w}}_{1,t} = \Phi_t \left(\widehat{W}_t^{\frac{1+\lambda_{\mathrm{L}}}{\lambda_{\mathrm{L}}}} \tilde{L}_t^{\mathrm{S}}\right)^{\nu} + \beta\mathscr{v}\mathbb{E}_t \frac{\Theta_{t+1}N_{t+1}}{\Theta_t N_t} \left(\frac{G_P}{G_{P,t+1}} \frac{G_W}{G_{W,t+1}}\right)^{-\frac{1+\lambda_{\mathrm{L}}}{\lambda_{\mathrm{L}}}} \frac{\tilde{L}_{t+1}^{\mathrm{S}}}{\tilde{L}_t^{\mathrm{S}}} \left(\frac{G_P}{G_{P,t+1}} \frac{G_W}{G_{A,t+1}^{\mathscr{a}} G_{A^*,t+1}^{\mathscr{e}}}\right)^{-\nu\frac{1+\lambda_{\mathrm{L}}}{\lambda_{\mathrm{L}}}} \widehat{\mathscr{w}}_{1,t+1}$, $\widehat{\mathscr{w}}_{2,t} = \widehat{\mathscr{m}}_t^{\mathrm{C}} + \beta\mathscr{v}\mathbb{E}_t \frac{\Theta_{t+1}N_{t+1}}{\Theta_t N_t} \left(\frac{G_P}{G_{P,t+1}} \frac{G_W}{G_{W,t+1}}\right)^{-\frac{1+\lambda_{\mathrm{L}}}{\lambda_{\mathrm{L}}}} \frac{\tilde{L}_{t+1}^{\mathrm{S}}}{\tilde{L}_t^{\mathrm{S}}} \frac{G_P}{G_{P,t+1}} \frac{G_W}{G_{A,t+1}^{\mathscr{a}} G_{A^*,t+1}^{\mathscr{e}}} \widehat{\mathscr{w}}_{2,t+1}$. This formulation avoids any explicit log-linearization and allows us to compute arbitrarily high order approximations to the model, for robustness checks. A similar formulation is used in Schmitt-Grohé and Uribe (2006).

- **Investment decisions:** for V $\in$ {P,R}:
$\frac{1}{E_t^{\mathrm{KV}}} = \Gamma_t \frac{\widehat{R}_t^{\mathrm{KV}}}{\delta^{\mathrm{V}\prime}(u_t^{\mathrm{V}})}\left[1 - Q^{\mathrm{KV}}(G_{I^{\mathrm{KV}*},t}) - G_{I^{\mathrm{KV}},t} Q^{\mathrm{KV}\prime}(G_{I^{\mathrm{KV}*},t})\right] + \beta \mathbb{E}_t \Xi_{t+1} \Gamma_{t+1} \frac{\widehat{R}_{t+1}^{\mathrm{KV}}}{G_{A^*,t}^{\xi_{\mathrm{KV}}} \delta^{\mathrm{V}\prime}(u_{t+1}^{\mathrm{V}})} G_{I^{\mathrm{KV}*},t+1}^2 Q^{\mathrm{KV}\prime}(G_{I^{\mathrm{KV}*},t+1})$, where $\widehat{R}_t^{\mathrm{KV}} := R_t^{\mathrm{KV}} A_t^{*\xi_{\mathrm{KV}}}$ and $G_{I^{\mathrm{KV}*},t} = G_{A^*,t}^{\xi_{\mathrm{KV}}} \frac{E_t^{\mathrm{KV}}}{E_{t-1}^{\mathrm{KV}}} G_{I^{\mathrm{KV}},t}$
- **Utilisation decisions:** for V $\in$ {P,R} : $\frac{\widehat{R}_t^{\mathrm{KV}}}{\delta^{\mathrm{V}\prime}(u_t^{\mathrm{V}})} = \beta \mathbb{E}_t \Xi_{t+1} \frac{\widehat{R}_{t+1}^{\mathrm{KV}}}{G_{A^*,t}^{\xi_{\mathrm{KV}}}}\left[u_{t+1}^{\mathrm{V}} + \frac{1-\delta^{\mathrm{V}}(u_{t+1}^{\mathrm{V}})}{\delta^{\mathrm{V}\prime}(u_{t+1}^{\mathrm{V}})}\right]$.
- **Capital accumulation:** for V $\in$ {P,R} : $\hat{K}_t^{\mathrm{V}} = \left(1 - \delta^{\mathrm{V}}(u_t^{\mathrm{V}})\right) \frac{\hat{K}_{t-1}^{\mathrm{V}}}{G_{N,t} G_{A,t}^{a} G_{A^*,t}^{e+\xi_{\mathrm{KV}}}} + \Gamma_t E_t^{\mathrm{KV}} \hat{I}_t^{\mathrm{KV}} [1 - Q^{\mathrm{KV}}(G_{I^{\mathrm{KV}},t})]$, where $\hat{K}_t^{\mathrm{V}} := \frac{K_t^{\mathrm{V}}}{N_t A_t^{a} A_t^{*e+\xi_{\mathrm{KV}}}}$ and $\hat{I}_t^{\mathrm{KV}} = \frac{I_t^{\mathrm{KV}}}{N_t A_t^{a} A_t^{*e}}$ (hence $G_{I^{\mathrm{KV}},t} = G_{N,t} G_{A,t}^{a} G_{A^*,t}^{e} \frac{\hat{I}_t^{\mathrm{KV}}}{\hat{I}_{t-1}^{\mathrm{KV}}}$).

5.2. Aggregate relationships

- **Aggregate mark-up pricing:** $\frac{\left[R_t^{\mathrm{KP}\,\alpha_{\mathrm{P}}} \widehat{W}_t^{\mathrm{EP}\,1-\alpha_{\mathrm{P}}}\right]^{1-\iota_{\mathrm{P}}}}{\iota_{\mathrm{P}}^{\iota_{\mathrm{P}}}(1-\iota_{\mathrm{P}})^{1-\iota_{\mathrm{P}}}[\alpha_{\mathrm{P}}^{\alpha_{\mathrm{P}}}(1-\alpha_{\mathrm{P}})^{1-\alpha_{\mathrm{P}}}]^{1-\iota_{\mathrm{P}}}} = \frac{1}{1+\mu_{t-1}}$ where $\widehat{W}_t^{\mathrm{EP}} := \frac{\widehat{W}_t}{E_t^{\mathrm{L}}\left[1-Q^{\mathrm{LP}}\left(\frac{\widehat{L}_t^{\mathrm{TP}}}{\widehat{L}_{t-1}^{\mathrm{TP}}} G_{N,t} G_{A^*,t}^{\xi_L}\right)\right]}$ and $\hat{L}_t^{\mathrm{TP}} = \frac{L_t^{\mathrm{TP}}}{N_t A_t^{*\xi_L}}$, where $L_t^{\mathrm{T}} := A_t^{*\xi_{\mathrm{L}}} E_t^{\mathrm{L}} L_t^{\mathrm{S}}$.
- **Mark-up aggregation:** $\left(\frac{1}{1+\mu_t}\right)^{\frac{1}{\lambda}} = \left(\frac{1}{1+\mu_t^{\mathrm{P}}}\right)^{\frac{1}{\lambda}} \mathcal{s}_t + \left(\frac{1}{1+\eta\lambda}\right)^{\frac{1}{\lambda}} (1 - \mathcal{s}_t)$, where $\mu_t^{\mathrm{P}} = \mu_t(I_t)$and $\mathcal{s}_t := (1 - \mathcal{q}) \frac{\mathcal{s}_{t-1}}{G_{I,t}} + 1 - \frac{1}{G_{I,t}}$.
- **Productivity aggregation:** $\left(\frac{\hat{A}_t}{1+\mu_{t-1}}\right)^{\frac{1}{\lambda}} = \left(\frac{1}{1+\mu_{t-1}^{\mathrm{P}}}\right)^{\frac{1}{\lambda}} \mathcal{s}_{t-1} + \left(\frac{\hat{A}_t^{\mathrm{N}}}{1+\eta\lambda}\right)^{\frac{1}{\lambda}} (1 - \mathcal{s}_{t-1})$, where $\hat{A}_t := \frac{A_t}{A_t^*}$ and $\hat{A}_t^{\mathrm{N}} := \left[\left(\frac{1}{G_{A^*,t}}\right)^{\frac{1}{\lambda}} \left(\frac{\mathcal{q}}{1/\mathcal{s}_{t-2}-(1-\mathcal{q})}\right) + \left(\frac{\hat{A}_{t-1}^{\mathrm{N}}}{G_{A^*,t}}\right)^{\frac{1}{\lambda}} \left(1 - \frac{\mathcal{q}}{1/\mathcal{s}_{t-2}-(1-\mathcal{q})}\right)\right]^{\lambda}$.

5.3. Firm decisions

- **Strategic in-industry pricing:** $\mu_t^{\mathrm{P}} = \lambda \frac{\eta \hat{J}_t^{\mathrm{P}}}{\hat{J}_t^{\mathrm{P}} - (1-\eta)}$, where $\hat{J}_t^{\mathrm{P}} = J_t(I_t)$.
- **Firm research decisions:** $\frac{\mathcal{d}_t}{\mathcal{p}\mu_t^{\mathrm{P}}} \mathbb{E}_t \Xi_{t+1} G_{Y,t+1} \hat{A}_{t+1}^{-\frac{1}{\lambda}} \frac{Z_{t+1} \hat{\mathcal{L}}_t^{\mathrm{R}}}{1+\gamma Z_{t+1} \hat{\mathcal{L}}_t^{\mathrm{R}}} = (1 - \mathcal{m}_t^{\mathrm{R}}) \mathbb{E}_t \Xi_{t+1} G_{Y,t+1} \hat{A}_{t+1}^{-\frac{1}{\lambda}}$, where $\hat{\mathcal{L}}_t^{\mathrm{R}} := A_t^{*-\zeta} X_t^{\mathrm{R}\,\iota_R} \left[K_t^{\mathrm{R}\,\alpha_R} L_t^{\mathrm{R}\,1-\alpha_R}\right]^{1-\iota_R}$ is the amount of effective research conducted by firms in protected industries.
- **Research and appropriation payoff:** $G_{A^*,t} = \left(1 + \gamma Z_t \hat{\mathcal{L}}_{t-1}^{\mathrm{R}}\right)^{\frac{1}{\gamma}}$.
- **Free entry of firms:** $\beta \frac{1}{\hat{I}_t \hat{J}_t^{\mathrm{P}}} \frac{\mu_t^{\mathrm{P}}}{1+\mu_t^{\mathrm{P}}} \left(\frac{1+\mu_t}{1+\mu_t^{\mathrm{P}}}\right)^{\frac{1}{\lambda}} \mathbb{E}_t \Xi_{t+1} G_{Y,t+1} \hat{A}_{t+1}^{-\frac{1}{\lambda}} = \frac{1}{\mathcal{p}} \hat{\mathcal{L}}_t^{\mathrm{R}} \frac{\hat{\mathcal{C}}_t}{\hat{Y}_t}$, where $\hat{I}_t := \frac{I_t}{N_t A_t^{a(1-\mathcal{b})} A_t^{*e-(c+\zeta)}}$ is the measure of products relative to its trend,[14]

[14] This means $G_{I,t} = G_{N,t} G_{A,t}^{a(1-\mathcal{b})} G_{A^*,t}^{e-(c+\zeta)} \frac{\hat{I}_t}{\hat{I}_{t-1}}$.

$\hat{Y}_t^{\mathrm{GROSS}} := \frac{Y_t^{\mathrm{GROSS}}}{N_t A_t^{\mathcal{a}} A_t^{*\mathcal{e}}}$ is *gross* output relative to trend and $\hat{\mathcal{C}}_t := \frac{\left[R_t^{\mathrm{KR}\,\alpha_{\mathrm{R}}} \widehat{W}_t^{\mathrm{ER}\,1-\alpha_{\mathrm{R}}}\right]^{1-\iota_{\mathrm{R}}}}{\iota_{\mathrm{R}}^{\iota_{\mathrm{R}}}(1-\iota_{\mathrm{R}})^{1-\iota_{\mathrm{R}}}\left[\alpha_{\mathrm{R}}^{\alpha_{\mathrm{R}}}(1-\alpha_{\mathrm{R}})^{1-\alpha_{\mathrm{R}}}\right]^{1-\iota_{\mathrm{R}}}}$ is the marginal cost of a unit of research or invention, divided by $A_t^{\mathcal{a}\mathcal{b}} A_t^{*\mathcal{c}}$ (where $\widehat{W}_t^{\mathrm{ER}} := \frac{\widehat{W}_t}{E_t^{\mathrm{L}}\left[1-Q^{\mathrm{LR}}\left(\frac{\hat{L}_t^{\mathrm{TR}}}{\hat{L}_{t-1}^{\mathrm{TR}}} G_{N,t} G_{A^*,t}^{\xi_L}\right)\right]}$ and $\hat{L}_t^{\mathrm{TR}} = \frac{L_t^{\mathrm{TR}}}{N_t A_t^{*\xi_L}}$).

5.4. Inventor decisions

- **Inventor profits:** are given recursively by:
  $\hat{\pi}_t = \frac{1-\wp}{\wp} \hat{\mathcal{L}}_t^{\mathrm{R}} \hat{\mathcal{C}}_t \hat{J}_t^{\mathrm{P}} + \beta(1-\mathcal{q})\mathbb{E}_t \Xi_{t+1} G_{A,t+1}^{\mathcal{a}\mathcal{b}} G_{A^*,t+1}^{\mathcal{c}+\zeta} \hat{\pi}_{t+1}$, where $\hat{\pi}_t := \frac{\pi_t}{A_t^{\mathcal{a}\mathcal{b}} A_t^{*\mathcal{c}+\zeta}}$.
- **Free entry of inventors:** Either $G_{I,t} \geq 1$ binds or $\mathcal{L}_t^{\mathrm{I}} \hat{\mathcal{C}}_t \geq \hat{\pi}_t$ does.

5.5. Market clearing

- **R&D expenditure:** $\mathrm{RND}_t := \hat{\mathcal{C}}_t \hat{I}_t \left[\mathcal{L}_t^{\mathrm{I}}\left(1-\frac{1}{G_{I,t}}\right) + \hat{\mathcal{L}}_t^{\mathrm{R}} \mathcal{s}_t \hat{J}_t^{\mathrm{P}}\right]$.
- **Labour market clearing:** $E_t^{\mathrm{L}} \hat{L}_t^{\mathrm{S}} = \hat{L}_t^{\mathrm{TY}} + \hat{L}_t^{\mathrm{TR}}$, where $\hat{L}_t^{\mathrm{S}} := \frac{L_t^{\mathrm{T}}}{A_t^{*\xi_{\mathrm{L}}} N_t E_t^{\mathrm{L}}}$.
- **Production labour market clearing:** $\widehat{W}_t \hat{L}_t^{\mathrm{TY}} = E_t^{\mathrm{L}}(1-\alpha_{\mathrm{P}})(1-\iota_{\mathrm{P}})\mathcal{J}_t \hat{Y}_t^{\mathrm{GROSS}}$ where $\mathcal{J}_t := \frac{\mathcal{s}_{t-1}}{1+\mu_{t-1}^{\mathrm{P}}}\left(\frac{1}{\hat{A}_t}\frac{1+\mu_{t-1}}{1+\mu_{t-1}^{\mathrm{P}}}\right)^{\frac{1}{\lambda}} + \frac{1-\mathcal{s}_{t-1}}{1+\eta\lambda}\left(\frac{\hat{A}_t^{\mathrm{N}}}{\hat{A}_t}\frac{1+\mu_{t-1}}{1+\eta\lambda}\right)^{\frac{1}{\lambda}}$ is a weighted measure of average inverse gross mark-ups.
- **R&D labour market clearing:** $\widehat{W}_t \hat{L}_t^{\mathrm{TR}} = E_t^{\mathrm{L}}(1-\alpha_{\mathrm{R}})(1-\iota_{\mathrm{R}})\mathrm{RND}_t$.
- **Capital markets clearing:** $u_t^{\mathrm{P}} \hat{K}_{t-1}^{\mathrm{P}} \widehat{R}_t^{\mathrm{KP}} = \alpha_{\mathrm{P}}(1-\iota_{\mathrm{P}})\mathcal{J}_t \hat{Y}_t^{\mathrm{GROSS}}$, $u_t^{\mathrm{R}} \hat{K}_{t-1}^{\mathrm{R}} \widehat{R}_t^{\mathrm{KR}} = \alpha_{\mathrm{R}}(1-\iota_{\mathrm{R}})\mathrm{RND}_t$
- **Goods market clearing:** $\hat{Y}_t = \hat{Y}_t^{\mathrm{GROSS}}(1-\iota_{\mathrm{P}}\mathcal{J}_t) - \iota_{\mathrm{R}}\mathrm{RND}_t - (1-\varrho_{\mathrm{GDP}})\hat{I}_t^{\mathrm{R}} = \hat{C}_t + \hat{I}_t^{\mathrm{P}} + \varrho_{\mathrm{GDP}}\hat{I}_t^{\mathrm{R}}$, where $\hat{Y}_t$ is GDP over $N_t A_t^{\mathcal{a}} A_t^{*\mathcal{e}}$ and ϱ_{GDP} specifies the proportion of R&D capital investment that is measured in GDP. (Given R&D itself is not measured in GDP it is not obvious that this equals 1.)
- **Monetary rule:**
  $$\frac{R_t^{\mathrm{NOM}}}{R^{\mathrm{NOM}}} = \left(\frac{R_t^{\mathrm{NOM}}}{R^{\mathrm{NOM}}}\right)^{\rho_{R^{\mathrm{NOM}}}} \left[\left(\frac{G_{P,t}}{G_{P,t}^*}\right)^{\mathcal{M}_{\mathrm{P}}} \left(\frac{E_{t-1}^{\mathrm{KP}} G_{A^*}^{\xi_{\mathrm{KP}}}}{E_t^{\mathrm{KP}} G_{A^*,t}^{\xi_{\mathrm{KP}}}}\right)^{\mathcal{M}_{\mathrm{PKP}}} \left(\frac{E_{t-1}^{\mathrm{KR}} G_{A^*}^{\xi_{\mathrm{KR}}}}{E_t^{\mathrm{KR}} G_{A^*,t}^{\xi_{\mathrm{KR}}}}\right)^{\mathcal{M}_{\mathrm{PKR}}} \left(\frac{G_{W,t}}{G_W}\right)^{\mathcal{M}_{\mathrm{W}}} \left(\frac{\widehat{R}_t^{\mathrm{KP}}}{\widehat{R}^{\mathrm{KP}}}\right)^{\mathcal{M}_{\mathrm{RKP}}} \cdot \left(\frac{\widehat{R}_t^{\mathrm{KR}}}{\widehat{R}^{\mathrm{KR}}}\right)^{\mathcal{M}_{\mathrm{RKR}}} \Theta_t^{\mathcal{M}_\Theta} \tilde{\delta}_t^{-\mathcal{M}_{\tilde{\delta}}}\right]^{1-\rho_{R^{\mathrm{NOM}}}} \cdot \left[\left(\frac{\hat{Y}_t}{\hat{Y}}\right)^{\mathcal{M}_{\mathrm{Y}}} \left(\frac{G_{Y,t}/G_{N,t}}{G_Y/G_N}\right)^{\mathcal{M}_{\mathrm{G}}}\right]^{1-\rho_{R^{\mathrm{NOM}}}} \exp \epsilon_{R^{\mathrm{NOM}},t}.$$

5.6. Observation equations

- **Nominal output growth:** $g_{Y,t} + g_{P,t} + \mathrm{me}_{\mathrm{Y},t} - \mathrm{me}_{\mathrm{Y},t-1}$, where $g_{Y,t} = \log\left(\frac{\hat{Y}_t}{\hat{Y}_{t-1}} G_{N,t} G_{A,t}^{a} G_{A^*,t}^{e}\right)$.
- **Consumption price inflation:** $g_{P,t} + \mathrm{me}_{\mathrm{PC},t} - \mathrm{me}_{\mathrm{PC},t-1}$.
- **Investment price inflation:** $g_{P,t} + g_{P^{\mathrm{I}},t} + \mathrm{me}_{\mathrm{PI},t} - \mathrm{me}_{\mathrm{PI},t-1}$, where:

$$G_{P^{\mathrm{I}},t} = \sqrt{\frac{\left(\frac{E_{t-1}^{\mathrm{KP}} \hat{I}_{t-1}^{\mathrm{KP}}}{E_t^{\mathrm{KP}} G_{A^*,t}^{\xi_{\mathrm{KP}}}} + \varrho_{\mathrm{GDP}} \frac{E_{t-1}^{\mathrm{KR}} \hat{I}_{t-1}^{\mathrm{KR}}}{E_t^{\mathrm{KR}} G_{A^*,t}^{\xi_{\mathrm{KR}}}}\right)}{\left(\hat{I}_{t-1}^{\mathrm{KP}} + \varrho_{\mathrm{GDP}} \hat{I}_{t-1}^{\mathrm{KR}}\right)} \frac{\left(\hat{I}_t^{\mathrm{KP}} + \varrho_{\mathrm{GDP}} \hat{I}_t^{\mathrm{KR}}\right)}{\left(\frac{E_t^{\mathrm{KP}} \hat{I}_t^{\mathrm{KP}} G_{A^*,t}^{\xi_{\mathrm{KP}}}}{E_{t-1}^{\mathrm{KP}}} + \varrho_{\mathrm{GDP}} \frac{E_t^{\mathrm{KR}} \hat{I}_t^{\mathrm{KR}} G_{A^*,t}^{\xi_{\mathrm{KR}}}}{E_{t-1}^{\mathrm{KR}}}\right)}}.$$

- **Population growth:** $g_{N,t} + \mathrm{me}_{\mathrm{N},t} - \mathrm{me}_{\mathrm{N},t-1}$.
- **Demeaned labour supply**: $l_t^{\mathrm{S}} + \mathrm{me}_{\mathrm{LS},t}$.
- **R&D share**: $\log\left(\frac{\mathrm{RND}_t + \varrho_{\mathrm{RND}} \hat{I}_t^R}{\hat{Y}_t}\right) + \mathrm{me}_{\mathrm{RND},t}$, where ϱ_{RND} is the proportion of R&D capital investment that is measured in the NIPA R&D measure. ($\varrho_{\mathrm{GDP}} + \varrho_{\mathrm{RND}} \leq 1$).
- **Consumption share:** $\log\left(\frac{\hat{C}_t}{\hat{Y}_t}\right) + \mathrm{me}_{\mathrm{C},t}$.
- **Labour share:** $\log\left(\frac{\widehat{W}_t \widehat{L}_t^{\mathrm{S}}}{\hat{Y}_t}\right) + \mathrm{me}_{\mathrm{L},t}$.
- **Depreciation share:** $\log\left(\frac{\delta^{\mathrm{Y}}(u_t^{\mathrm{Y}}) \widehat{K}_{t-1}^{\mathrm{Y}}}{\hat{Y}_t \left(G_{N,t} G_{A,t}^{a} G_{A^*,t}^{e+\xi_{\mathrm{KY}}} E_t^{\mathrm{KY}}\right)} + \varrho_{\mathrm{GDP}} \frac{\delta^{\mathrm{R}}(u_t^{\mathrm{R}}) \widehat{K}_{t-1}^{\mathrm{R}}}{\hat{Y}_t \left(G_{N,t} G_{A,t}^{a} G_{A^*,t}^{e+\xi_{\mathrm{KR}}} E_t^{\mathrm{KR}}\right)}\right) + \mathrm{me}_{\mathrm{D},t}$.
- **Demeaned nominal interest rates:** $\log\left(\frac{R_t^{\mathrm{NOM}}}{R^{\mathrm{NOM}}}\right) + \mathrm{me}_{\mathrm{R},t}$.
- **Capacity utilisation:** $\frac{u_t^{\mathrm{Y}} \frac{\widehat{K}_{t-1}^{\mathrm{Y}}}{G_{A^*,t}^{\xi_{\mathrm{KY}}} E_t^{\mathrm{KY}}} + u_t^{\mathrm{R}} \varrho_{\mathrm{GDP}} \frac{\widehat{K}_{t-1}^{\mathrm{R}}}{G_{A^*,t}^{\xi_{\mathrm{KR}}} E_t^{\mathrm{KR}}}}{\frac{\widehat{K}_{t-1}^{\mathrm{Y}}}{G_{A^*,t}^{\xi_{\mathrm{KY}}} E_t^{\mathrm{KY}}} + \varrho_{\mathrm{GDP}} \frac{\widehat{K}_{t-1}^{\mathrm{R}}}{G_{A^*,t}^{\xi_{\mathrm{KR}}} E_t^{\mathrm{KR}}}} + \mathrm{me}_{\mathrm{U},t}$. (The capital stocks enter here in order to correctly weight to produce the average utilisation.)
- **BAA-AAA Spread:** $\varsigma_0 - \varsigma_1 \log \Gamma_t + \mathrm{me}_{\mathrm{S},t}$.

6. Data details

- **Nominal output growth** (1947Q2 – 2011Q2), from NIPA table 1.1.5.
- **Consumption price inflation** (1947Q2 – 2011Q2), including non-durables and durables (from NIPA table 1.1.4) and government consumption[15] (from

[15] We are implicitly making the optimistic assumption that government consumption is a perfect substitute for private consumption. This is a simplifying shortcut to save us modelling government consumption.

NIPA table 3.9.4) and excluding education[16] (from NIPA tables 2.4.4[17] and 3.15.4[18]).

- **Investment price inflation** (1947Q2 – 2011Q2), including education (data sources as for consumption price inflation).
- **Population growth** (1948Q2 – 2011Q2), X-12 seasonally adjusted, from the BLS's Civilian Non-institutional Population Over 16 series.
- **Labour supply per capita** (1948Q1 – 2011Q2), from NIPA table 6.9, interpolated to quarterly using the Litterman (1983) method, with "Business Sector: Hours of All Persons" from the BEA as a high frequency indicator.
- **R&D share** (1959Q1 – 2007Q4), given by R&D expenditure from NIPA R&D Satellite Account (1959-2007) table 2.1, over GDP from NIPA table 1.1.5, interpolated to quarterly using the Litterman (1983) method with GDP as the high frequency indicator.
- **Consumption share** (1947Q1 – 2011Q2), given by consumption of durables and non-durables (from NIPA table 1.1.5) plus government consumption (from NIPA table 3.9.5) minus education expenditure (from NIPA table 2.4.5[19] and NIPA table 3.15.5[20]) all over GDP (from NIPA table 1.1.5).[21]
- **Labour share** (1947Q1 – 2011Q2), given by compensation of employees paid from NIPA table 1.10, over GDP (from NIPA table 1.1.5).

[16] Removing education from the consumption share brings it substantially closer to stationarity, so it is important to do the same for the price level too. The price disaggregation necessary to remove education was performed by inverting the Fisher formula, which, due to its approximate aggregation property (Diewert 1978) is sufficiently accurate.

[17] Interpolated to quarterly using the Litterman (1983) method, with consumption and investment prices as indicators (from NIPA table 1.1.4).

[18] Extrapolated back to 1947 using the Litterman (1983) method, with government consumption and investment prices (from NIPA table 3.9.4) and private education prices (from NIPA table 2.4.4) as indicators, then interpolated to quarterly using the same method with government consumption and investment prices (from NIPA table 3.9.4) as high frequency indicators.

[19] Interpolated to quarterly using the Litterman (1983) method, with consumption and investment as indicators (from NIPA table 1.1.5).

[20] Extrapolated back to 1947 using the Litterman (1983) method with log-linearly interpolated data from the National Centre for Education Statistics, Digest of Education Statistics 2010, table 29 as an indicator, along with government consumption and investment (from NIPA table 3.9.5) and private education expenditure (from NIPA table 2.4.5). Then interpolated using the same method with government consumption and investment (from NIPA table 3.9.5) as high frequency indicators.

[21] In fitting this to the model, we are implicitly treating net exports as investment.

- **Depreciation share** (1947Q1 – 2011Q2), given by consumption of fixed capital from NIPA table 1.10, over GDP (from NIPA table 1.1.5).
- **Nominal interest rates** (1947Q1 – 2011Q2), in particular, the 3-month Treasury bill secondary market rate, from the FRB, release H.15.
- **Capacity utilisation** (1967Q1 – 2011Q2), (total industry) from the FRB, release G.17, table 7.
- **BAA-AAA Spread** (1947Q1 – 2011Q2), from the FRB, release H.15.

7. Estimated parameters

Any parameters in bold are fixed rather than estimated. All values are reported to three significant figures, except those below 10^{-4} which are rounded down to zero, those which are of the form $1+x$, with $|x|<0.1$ in which case we give x to three significant figures, percentages, which are given to one decimal place, and approximate standard errors (in brackets) which are given to two significant figures.

Variable	**Value**	**Variable**	**Value**
ν	0.250 (0.0056)	β	**0.99**
$\mathscr{h}$	0.253 (0.0041)	$\mathscr{h}^{\mathrm{INT}}$	0.0151 (0.0032)
$\mathscr{h}^{\mathrm{LS}}$	0 (0)	$\mathscr{v}$	0.826 (0.0042)
λ	0.320 (0.00054)	λ_{L}	0.170 (0.0041)
$\mathscr{p}$	0.0427 (0.00021)	$\mathscr{q}$	0.0374 (0.00030)
$\rho_{R^{\mathrm{NOM}}}$	0.615 (0.013)	$\mathcal{M}_{\mathrm{P}}$	1.0275 (0.0059)
$\mathcal{M}_{\mathrm{PKP}}$	0 (0)	$\mathcal{M}_{\mathrm{PKR}}$	0 (0)
$\mathcal{M}_{\mathrm{RKP}}$	0.0509 (0.0016)	$\mathcal{M}_{\mathrm{RKR}}$	0 (0)
$\mathcal{M}_{\Theta}$	0 (0)	$\mathcal{M}_{\tilde{\delta}}$	0.0108 (0.0074)
$\mathcal{M}_{\mathrm{Y}}$	0 (0)	$\mathcal{M}_{\mathrm{G}}$	0 (0)
$\mathcal{M}_{\mathrm{W}}$	0 (0)		
$\exp \varsigma_0$	2.57 (2.9×10^{-5})	ς_1	872 (880)
ϱ_{GDP}	0.494 (0.013)	ϱ_{RND}	0.506 (0.013)
ζ	0 (0)	ξ_{L}	0.0859 (0.0012)
ξ_{KP}	0.0828 (0.00053)	ξ_{KR}	2.73 (0.0094)
α_{P}	0.201 (0.00040)	α_{R}	0.996 (7.4×10^{-6})
ι_{P}	0.0427 (0.0011)	ι_{R}	0.178 (0.0032)
$\delta^{\mathrm{P}}(u^{\mathrm{P}})$	0.0189 (7.5×10^{-5})	$\delta^{\mathrm{R}}(u^{\mathrm{R}})$	0.0284 (0.00062)
$\delta^{\mathrm{P}\prime}(u^{\mathrm{P}})$	0.0413 (0.00011)	$\delta^{\mathrm{R}\prime}(u^{\mathrm{R}})$	0.0501 (0.00063)
$\delta^{\mathrm{P}\prime\prime}(u^{\mathrm{P}})$	1.64 (0.035)	$\delta^{\mathrm{R}\prime\prime}(u^{\mathrm{R}})$	133 (9.4)
$\frac{d}{d\tilde{\delta}}\log \delta^{\mathrm{P}}(u^{\mathrm{P}})$	**1**	$\frac{d}{d\tilde{\delta}}\log \delta^{\mathrm{R}}(u^{\mathrm{R}})$	64.2 (1.5)
$\frac{d}{d\tilde{\delta}}\log \delta^{\mathrm{P}\prime}(u^{\mathrm{P}})$	64.2 (1.5)	$\frac{d}{d\tilde{\delta}}\log \delta^{\mathrm{R}\prime}(u^{\mathrm{R}})$	0 (0)
$Q^{\mathrm{P}\prime\prime}(G_{I^{\mathrm{KP}*}})$	0.00533 (0.0012)	$Q^{\mathrm{R}\prime\prime}(G_{I^{\mathrm{KR}*}})$	62.6 (4.0)
$Q^{\mathrm{LP}\prime}(G_{L^{\mathrm{TP}}})$	0.0875 (0.0047)	$Q^{\mathrm{LR}\prime}(G_{L^{\mathrm{TR}}})$	0 (0)

Table 1: Estimated parameters, excluding shocks.

Variable	V (i.e. steady-state)	ρ_V	$100\sigma_V$	p-value on 1 lag LM-test[22]
Φ	1.0349 (0.0047)	0.815 (0.010)	2.46 (0.16)	0
Θ	**1**	0.443 (0.0056)	0.0231 (0.0114)	**0.0318**
G_{N}	1.00372 (1.4×10^{-5})	0.0675 (0.019)	0.103 (0.0021)	**0.146**
$\mathcal{L}^{\mathrm{I}}$	7.26 (0.034)	0 (0)	0 (0)	0
Z	**1**	**0**	0 (0)	0
Γ	**1**	0 (0)	0 (0)	0.000926
E^{L}	**1**	0.614 (0.0056)	0 (0)	**0.725**
E^{KP}	**1**	0 (0)	0 (0)	0
E^{KR}	**1**	0.664 (0.0071)	0.000360 (0.00012)	0.000148
$G^{*}_{P,t}$	1.00851 (6.1×10^{-6})	0.887 (0.00027)	0 (0)	**0.161**
η	0.169 (0.00024)	0.0605 (0.2)	0.0147 (0.012)	0
γ	18.6 (0.054)	0 (0)	0 (0)	0
$\exp\tilde{\delta}$	**1**	0.862 (0.0027)	0.403 (0.011)	**0.958**
R^{SHOCK}_t	**1**	**0**	0.00824 (0.00075)	**0.464**

Table 2: Estimated parameters from non-m.e. shocks, tests of misspecification of their residuals.

Each shock takes the form $\log V_t = (1 - \rho_V) \log V + \rho_V \log V_{t-1} + \sigma_V \epsilon_{V,t}$, where $\epsilon_{v,t}$~NIID$(0, 1)$.

Variable	Φ	Θ	G_{N}	$\mathcal{L}^{\mathrm{I}}$	Z	Γ	E^{L}	E^{KP}	E^{KR}	$G^{*}_{P,t}$	η	γ	R^{SHOCK}_t	$\exp\tilde{\delta}$
Nom. output growth	**17.9**	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	**81.1**
Con. price inflation	**37.5**	0.0	**2.5**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.4	**59.6**
Inv. price inflation	**37.1**	0.0	**2.4**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.4	**60.1**
Population growth	0.0	0.0	**100.0**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Lab. supply per capita	**60.4**	0.0	0.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	**38.8**
R&D share	**2.1**	0.0	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**97.6**
Consumption share	**45.0**	0.0	0.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.3	**54.3**
Labour share	**1.8**	0.0	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**97.9**
Depreciation share	**45.3**	0.0	**1.8**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	**52.7**
Nominal interest rates	**41.5**	0.0	**2.0**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	**56.4**
Capacity utilisation	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**99.9**
BAA-AAA Spread	0.0	0.0	0.0	0.0	0.0	**100.0**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 3: Percentage non-m.e. variance decomposition of the observation variables.[23]

[22] Bold values indicate the cases in which we cannot reject the null hypothesis of no auto-correlation at 1%. The test uses heteroskedasticity robust standard errors. The lag length of 1 was preferred by the AIC, AICc and BIC criterions for all variables.

[23] Bold values are larger than 1%.

8. Additional figures

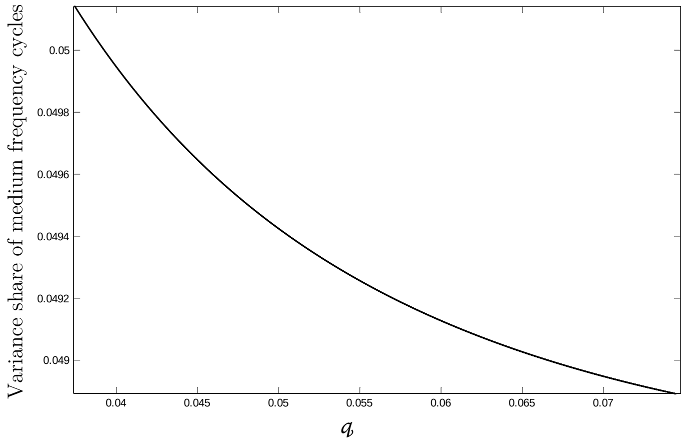

Figure 1: The effect of patent duration on the importance of medium-frequency cycles.

9. Granger (1969) causality tests in the presence of measurement error

Here we describe two experiments, both of which reveal that Granger (1969) causality tests may point in the opposite direction to the model's information flow, when the true leading series is subject to an additional idiosyncratic shock ("measurement error").

In our first set up, we modelled x_t as a repeated-root AR(2) process given by $x_t = 2\rho x_{t-1} - \rho^2 x_{t-2} + \sigma_x \varepsilon_{x,t}$, with $\varepsilon_{x,t}$~NIID(0,1), and σ_x chosen to ensure $\operatorname{Var} x_t = 1$. Then we defined $y_t = x_t + \varepsilon_{y,t}$ and $z_t = x_{t-1} + \frac{1}{100}\varepsilon_{z,t}$, with $\varepsilon_{y,t}, \varepsilon_{z,t}$~NIID(0,1). Note that the "measurement-error" in y_t is much-larger than that in z_t, but that conditional on $\varepsilon_{y,t}$, y_t contains information useful for forecasting z_t, namely, x_t. y_t here is playing the role of mark-ups in our empirical paper (Holden 2013b), and z_t is playing the role of output.

We then simulated 400 periods from this model, and ran Granger causality tests[24] at one to ten lags on the pairs (x_t, z_t) and (y_t, z_t). As expected, at all lags x_t was found to Granger cause z_t, with reverse causation only with a single lag. However, at all lags z_t was found to Granger cause y_t (again with reverse causation only with a single lag). Thus, the mere fact of adding the i.i.d. "measurement-error" $\varepsilon_{y,t}$ to x_t has reversed the direction of Granger causality.

[24] Code for this was taken from Seth et al. (2010).

This is because z_t contains valuable information about the expectation of future y_t, due to the persistence of the underlying process.

In a second experiment, we attempted to capture accurately the cross-correlation of output and mark-ups with a deterministic cycle. In particular, we instead set $x_t = \sin\left(\frac{\pi}{10}(t+4)\right)$, $y_t = x_t + \varepsilon_{y,t}$ and $z_t = \sin(\frac{\pi}{10}t) + \frac{1}{100}\varepsilon_{z,t}$, with $\varepsilon_{y,t}, \varepsilon_{z,t}$~NIID(0,1), and the interpretation of variables as before. Under this set-up, the cross correlation of y_t and z_t looks very similar to that between mark-ups and output. After simulating and running Granger causality tests as before, we found that at all lags x_t Granger causes z_t (with reverse causation only with a single lag), and that at all lags except the first z_t Granger causes y_t (with reverse causation only with a single lag). Once again then, the presence of measurement error in the leading variable is sufficient to reverse the direction of Granger causality.

We conclude then that Granger causality tests are unreliable in the presence of idiosyncratic noise or measurement error.

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