Learning from learners

Tom Holden[1], School of Economics, University of Surrey

Abstract: *Traditional macroeconomic learning algorithms are misspecified when all agents are learning simultaneously. In this paper, we produce a number of learning algorithms that do not share this failing, and show that this enables them to learn almost any solution, for any parameters, implying learning cannot be used for equilibrium selection. As a by-product, we are able to show that when all agents are learning by traditional methods, all deep structural parameters of standard new-Keynesian models are identified, overturning a key result of Cochrane (2009; 2011). This holds irrespective of whether the central bank is following the Taylor principle, irrespective of whether the implied path is or is not explosive, and irrespective of whether agents' beliefs converge. If shocks are observed then this result is trivial, so following Cochrane (2009) our analysis is carried out in the more plausible case in which agents do not observe shocks.*

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[1] Contact address: School of Economics, FBEL, University of Surrey, Guildford, GU2 7XH, England
Contact email: thomas.holden@gmail.com Web-site: http://www.tholden.org/
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1. Introduction

There is a contradiction at the heart of the traditional approach to macroeconomic learning (Marcet and Sargent (1989), Evans and Honkapohja (2001)). In this literature, each of the agents in an economy is supposed to run a regression that is correctly specified when all the other agents know the true law of motion. Were it indeed the case that only one agent in the economy had partial information about the economy's law of motion, then this agent's regression would always converge to the true law of motion, meaning that "learnability" in this weak sense is of no use for equilibrium selection. The literature supposes instead that all agents are learning at the same time, yet they continue to run a regression that is only correctly specified when everyone else has full information. As a result, these agents would be readily able to detect the misspecification in their regression, through evidence of serially correlated errors, or parameter non-constancy. This misspecification is most clear precisely when learning fails, meaning a finding of non-learnability via the traditional method only implies that agents would switch from that traditional method to a more sophisticated one. In this paper, we demonstrate the existence of a family of learning mechanisms that remain correctly specified when all agents are learning simultaneously.

Along the way, we will answer three challenges raised by Cochrane (2009) (directly or otherwise). Firstly, we will show that the non-observability of shocks does not pose any fundamental challenges either to learning, or to the formation of rational expectations, and we give general conditions under which a rational expectations equilibrium is precisely implementable without observing shocks.[2] Secondly, we show that serially correlated monetary policy shocks do not prevent Taylor-rule parameter identification, at least when everyone is learning at the same time, whether or not the central bank is following active policy. Finally, we demonstrate a learning mechanism capable of learning stationary minimal state variable (McCallum 1983) solutions whenever they exist, and another that may converge towards any sunspot solution, including explosive ones,[3] though a simple extension of our mechanism will rule out the latter when (and only when) they are prohibited by transversality or non-explosiveness constraints. Since, new-Keynesian models generally have no such constraints ruling out explosive paths for inflation (Cochrane 2011), in such models there is no guarantee that the stationary minimal state variable solution will be learnt, meaning that Cochrane (2009) was correct to conclude that learnability could not "save" the standard logic of new-Keynesian models.

The structure of our paper is as follows. In section 2 and the first appendix (7.1), we derive the general solution of a rational expectations model, under determinacy and indeterminacy, when shocks are unobserved. The resulting reduced form solution will be the basis of all of the learning mechanisms considered. The presence of sunspot shocks in the general solution will be key to our proof of structural parameter identification when agents are learning. In section 3, we show that an

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[2] In general a Kalman filter must be used as in Pearlman, Currie, and Levine (1986) or Ellison and Pearlman (2011), and impulse responses will differ.

[3] We cannot guarantee asymptotic convergence to explosive solutions, nonetheless beliefs will at least initially approach these solutions, and they will certainly diverge from beliefs under the stationary minimal state variable solution.

awareness that everyone else is learning is sufficient to achieve identification even when other agents are learning using a traditional method. Then in section 4, we introduce our family of sophisticated learning algorithms under which everyone in the economy realises everyone else is learning at the same time.

2. FREE solutions

2.1. Motivating example

Suppose, following Cochrane (2009), that the central bank follows the Taylor rule:

$$i_t = r + \frac{1}{\beta}(x_t - \gamma - \sigma s_t), \tag{2.1}$$

where x_t is the inflation rate, r is the constant real interest rate, $\frac{\gamma}{1-\beta}$ is the inflation target and s_t is the monetary policy shock which is given by:

$$s_t = \rho s_{t-1} + \varepsilon_t,$$

with $\varepsilon_t \sim \mathrm{NIID}(0,1)$. From the Fisher equation, we also have that:

$$i_t = r + \mathbb{E}_t x_{t+1}.^4 \tag{2.2}$$

Hence, from combining (2.1) and (2.2):

$$x_t = \beta \mathbb{E}_t x_{t+1} + \gamma + \sigma s_t.$$

More generally, there might also be a lag term in the model. Here, this would emerge if the central bank used the rule:

$$i_t = r + \frac{1}{\beta}\big((1-\alpha)x_t + \alpha \Delta x_t - \gamma - \sigma s_t\big)$$

which punishes accelerating inflation, and leads to the general univariate model:

$$x_t = \alpha x_{t-1} + \beta \mathbb{E}_t x_{t+1} + \gamma + \sigma s_t. \tag{2.3}$$

We work with this general model not because we believe central banks respond to inflation acceleration, but because in its multivariate version this structure encompasses all linear macroeconomic models, and we wish to make clear nothing we say is specific to the $\alpha = 0$ case.

The crucial thing to note about (2.3) is that since the transversality conditions of the consumer's optimisation problem do not restrict inflation, when solving this model there is no justification for restricting ourselves to stationary solutions.[5]

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[4] Throughout this document, variables with t subscripts are in the information set under which $\mathbb{E}_t$ is taken.

[5] If the Taylor rule is the result of optimal policy on behalf of the central bank, then there will in general be a transversality constraint coming from the central bank's optimisation problem that restricts inflation. But since it is consumer inflation expectations that determine the solution picked, the central bank's transversality constraint does not rule out explosive solutions, conditional on them using a Taylor rule.

2.2. Solution

For the time being, we suppose that all the agents in the economy have full knowledge of the values of α, β, γ, ρ and σ, and may observe x_t (and its lags), and $\mathbb{E}_{t-1}x_t$ (and its lags), at t. In our motivating example, the observability of expectations just requires nominal interest rates to be observable, thanks to the constant real interest rate, and the Fisher equation, (2.2). In reality, expectations may still be observed thanks to the survey of professional forecasters (or, more plausibly, media reports based on economic pundit's expectations). Expectations are also effectively observable if agents have access to prices from futures markets, or if they know that all other agents are forming expectations via the same mechanism. The traditional learning literature usually assumes homogeneous beliefs across agents, and we will continue to do so here, so in the models we work with, even in the absence of observable nominal and real interest rates, or observable futures contracts, aggregate expectations will always be observable.

We do not assume however that agents may observe s_t or ε_t. As pointed out by Cochrane (2009), that most shocks in DSGE models should be observable is rather implausible, thus ruling out rational expectations equilibria (REE) which require the observability of shocks seems like a minimal sensible restriction. We call the set of resulting equilibria the feasible rational expectations equilibria (FREE) of the original model. The key trick that enables agents to form expectations without seeing shocks is the fact that current news about past expectational errors is informative about the current shock. Thus, in general, agents will form expectations as a linear function of their lagged expectations.

To see this, let us begin by defining the expectational error by $\eta_t \coloneqq x_t - \mathbb{E}_{t-1}x_t$. Now, normally when solving rational expectations models we choose η_t to rule out explosive solutions, but here this is not justified, due to the lack of a consumer transversality condition on inflation. Thus there is a REE to the model for any η_t satisfying $\mathbb{E}_{t-1}\eta_t = 0$. Without loss of generality then, we may assume (following Lubik and Schorfheide (2003)) that $\eta_t = m_{\varepsilon,t-1}\varepsilon_t + m'_{\zeta,t-1}\zeta_t$, for some sunspot shock ζ_t (possibly a vector) satisfying $\mathbb{E}_{t-1}\zeta_t = 0$, $\mathbb{E}_{t-1}\varepsilon_t\zeta_t = 0$ and $\mathbb{E}_{t-1}\zeta_t\zeta'_t = I$, and some possibly time-varying belief parameters $m_{\varepsilon,t-1}$ and $m_{\zeta,t-1}$, known at $t-1$. (There is no reason why agents should always believe in the same set of sunspot shocks.)

Under the assumption then that $m_{\varepsilon,t-1} \neq 0$ for all t, subtracting ρ times the first lag of (2.3) from (2.3), gives:

$$\begin{aligned} x_t &= (\alpha+\rho)x_{t-1} - \alpha\rho x_{t-2} + \beta\mathbb{E}_t x_{t+1} - \beta\rho\mathbb{E}_{t-1}x_t + (1-\rho)\gamma + \sigma\varepsilon_t \\ &= (\alpha+\rho)x_{t-1} - \alpha\rho x_{t-2} + \beta\mathbb{E}_t x_{t+1} - \beta\rho\mathbb{E}_{t-1}x_t + (1-\rho)\gamma \\ &\qquad + \sigma\frac{x_t - \mathbb{E}_{t-1}x_t - m'_{\zeta,t-1}\zeta_t}{m_{\varepsilon,t-1}}. \end{aligned} \tag{2.4}$$

Hence providing $\beta \neq 0$[6]:

$$\begin{aligned}\mathbb{E}_t x_{t+1} = \frac{1}{\beta}\left(1-\frac{\sigma}{m_{\varepsilon,t-1}}\right)x_t - \frac{1}{\beta}(\alpha+\rho)x_{t-1} + \frac{1}{\beta}\alpha\rho x_{t-2} + \left(\rho+\frac{1}{\beta}\frac{\sigma}{m_{\varepsilon,t-1}}\right)\mathbb{E}_{t-1}x_t \\ - \frac{1}{\beta}(1-\rho)\gamma + \frac{1}{\beta}\frac{\sigma}{m_{\varepsilon,t-1}} m'_{\zeta,t-1}\zeta_t,\end{aligned} \tag{2.5}$$

which enables agents to form rational expectations without observing the value of shocks (i.e. s_t or ε_t). Thus providing $\beta \neq 0$, almost all of the model's REE are FREE.

When $|\alpha+\beta| < 1$, the unique stationary minimal state variable (MSV) solution corresponds to setting $m_{\varepsilon,t} \equiv m_\varepsilon^{\mathrm{MSV}} \coloneqq \sigma\left[\frac{1}{2} - \beta\rho + \frac{1}{2}\sqrt{1-4\alpha\beta}\right]^{-1}$ and $m_{\zeta,t} \equiv m_\zeta^{\mathrm{MSV}} \coloneqq 0$. To see this, let us first define:

$$v_t \coloneqq \mathbb{E}_t x_{t+1} - a_1^{\mathrm{MSV}} x_t - a_2^{\mathrm{MSV}} x_{t-1} - c^{\mathrm{MSV}}$$

where $a_1^{\mathrm{MSV}} \coloneqq \rho + \frac{1-\sqrt{1-4\alpha\beta}}{2\beta}$, $a_2^{\mathrm{MSV}} \coloneqq -\rho\frac{1-\sqrt{1-4\alpha\beta}}{2\beta}$ and $c^{\mathrm{MSV}} \coloneqq \frac{2(1-\rho)\gamma}{1-2\beta+\sqrt{1-4\alpha\beta}}$. Hence, $\mathbb{E}_t x_{t+1} = a_1^{\mathrm{MSV}} x_t + a_2^{\mathrm{MSV}} x_{t-1} + c^{\mathrm{MSV}} + v_t$ for all t. Then, when $m_{\varepsilon,t} \equiv m_\varepsilon^{\mathrm{MSV}}$ and $m_{\zeta,t} \equiv m_\zeta^{\mathrm{MSV}}$, from (2.5):

$$\mathbb{E}_t x_{t+1} = a_1^{\mathrm{MSV}} x_t + a_2^{\mathrm{MSV}} x_{t-1} + c^{\mathrm{MSV}} + \frac{1+\sqrt{1-4\alpha\beta}}{2\beta} v_{t-1},$$

i.e. $v_t = \frac{1+\sqrt{1-4\alpha\beta}}{2\beta} v_{t-1}$. Now when $|\alpha+\beta| < 1$ and $\alpha\beta < {}^1\!/_4$ (so x_t is real), $\frac{1+\sqrt{1-4\alpha\beta}}{2\beta} > 1$, therefore x_t is stationary if and only if $v_t = 0$ for all t, i.e. if and only if expectations always take this minimum state variable form. However, since current expectations are not constrained to render past expectations rational, if agents find themselves off the $v_t = 0$ path, it is still rational for them to jump back onto it, at least if x_t is constrained to be stationary.

Linear models such as this have two MSV solutions, however only one of them will be stationary under determinacy. In the below we refer to the MSV solution that is stationary under determinacy as the SMSV solution.

2.3. Generalization

All our analysis in the body of this paper will be confined to the univariate case; however, the tricks used above to express expectations as a function of observables carry over to the multivariate case, and the case in which at least some combinations of variables are constrained by transversality. This is discussed in the first appendix, section 7.1, where we provide a range of necessary and/or sufficient conditions for the existence of FREE solutions in multivariate models. Particularly intuitive results include the facts that:

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[6] Automatic in the particular case under consideration, but in other models there may be particular parameters for which expectations cease to matter, and in the multivariate case, β may not be invertible.

- if the model is completely indeterminate (perhaps because of a lack of transversality conditions), so there are as many degrees of freedom in expectations as there are variables, and there are at most as many shocks as variables, then almost all REE are FREE;
- there is always a REE with the form $\mathbb{E}_t x_{t+1} = T_{-1,21} x_{t-1} + T_{-1,22} \mathbb{E}_{t-1} x_t + T_{\mu,2} + T_{s,2} s_t$, which is always a FREE when $\dim s_t = 1$, and is a FREE more generally providing:
  - $T_{s,2}$ has linearly independent columns,
  - the number of explosive (or transversality violating) roots is greater or equal to $\dim s_t$,
  - a further technical condition is satisfied;
- if the unobserved shocks are not serially correlated, and if for any linear combination of shocks which does not appear in the transversality-violating block, that same linear combination does not appear anywhere in the model (i.e. agents can back out the value of relevant shocks from observing jump variables), then the model has at least one FREE, and a continuum under indeterminacy.

In all cases, the FREE solution to the model takes the form:

$$\mathbb{E}_t x_{t+1} = \mathcal{A}_1 x_t + \mathcal{A}_2 x_{t-1} + \mathcal{A}_3 x_{t-2} + \mathcal{B}_1 \mathbb{E}_{t-1} x_t + \mathcal{B}_2 \mathbb{E}_{t-2} x_{t-1} + c + d'_{1,t-1} \zeta_t,$$

which is identical to the univariate case, except for the extra lag on expectations.

These results hopefully go some way to reassuring the reader that although from here on in we will be focussing on the univariate case, the non-observability of shocks does not cause any additional problems when we generalise to the multivariate case.[7]

3. Learning (and identifying) from unsophisticated learners

We now turn to the formation of expectations when the values of α, β, γ, ρ, σ, $m_{\varepsilon,t}$ and $m_{\zeta,t}$ are not common knowledge. Before introducing our misspecification free learning methods in section 4, we address the issue of parameter identification when the agents in an economy are using a traditional learning method. For the duration of this section, we also assume it is common knowledge that $m_{\varepsilon,t}$ and $m_{\zeta,t}$ are constant across time, since the traditional models of macroeconomic learning cannot deal with actual laws of motion (ALMs) with time varying parameters.

[7] These results are closely related to the conditions derived by Levine et al. (2012) for solutions under imperfect information to be identical to solutions under perfect information. The results of Levine et al. (2012) are at once more general than our results (as they allow for arbitrary informational assumptions, rather than assuming that only shocks are unobserved) and less general (as they are restricted to the solutions of determinate models, and depend on assorted strong invertability assumptions).

3.1. Set-up

Under the saddle-path learning method of Ellison and Pearlman (2011), agents learn using the same rule they use to form expectations. Under the FREE solution to (2.3), given in equation (2.5), this suggests that agents should learn by estimating the regression model:

$$x_{t+1} = a_1 x_t + a_2 x_{t-1} + a_3 x_{t-2} + b\mathbb{E}^*_{t-1} x_t + c + d_1' \zeta_t + \eta_{t+1}, \\ \eta_{t+1} \sim \mathrm{NIID}\left(0, \sigma_\eta^2\right), \tag{3.1}$$

where $\mathbb{E}^*_{t-1} x_t$ is lagged aggregate (not-necessarily rational) expectations, which are observable for the reasons given previously.

If agents observed shocks, then by replacing η_{t+1} with $m_\varepsilon \varepsilon_{t+1} + m_\zeta' \zeta_{t+1}$, this would become an exact line fitting exercise, rather than a regression problem: after a finite number of periods agents would know the value of all parameters, thanks to the observability of $\mathbb{E}^*_{t-1} x_t$. (We also need that there is at least some variation in $\mathbb{E}^*_{t-1} x_t$ that is independent of the other terms, this will be true providing initial beliefs about a_3 and/or d_1 are non-zero.) Thus when shocks are observed, learning is trivial. This further justifies our focus on the non-observable shock case in this paper.

3.2. (Non-)Identification via OLS

Given that it is common knowledge that $m_{\varepsilon,t}$ and $m_{\zeta,t}$ are constant, the "true" model has $6 + \dim \zeta_t$ free parameters (α, β, γ, ρ, σ, m_ε, m_ζ), and by running the regression (3.1) agents will also learn $6 + \dim \zeta_t$ parameters (a_1, a_2, a_3, b, c, σ_η^2, d_1), which is a necessary condition for the identification of all of the model's parameters. This also means that if any variables are omitted from this regression (as they are in the traditional regressions used in the literature) then agents will have no information about at least one of the model's parameters.

Providing $\rho \neq 1$ and $\sigma > 0$, equating terms reveals that all the model's parameters are uniquely identified if any only if either $\alpha = \rho = 0$, or the following equation for β has a unique solution:[8]

$$\beta^3 a_3 = \left(-\beta^2 a_2 - (\beta b - 1 + \beta a_1)\right)(\beta b - 1 + \beta a_1).$$

Tedious algebra reveals that this in turn holds if any only if $\alpha \neq 0$, $\rho \neq 0$ and $\alpha\beta > \frac{1}{4}$, which implies there is no non-explosive, real, minimal state variable solution for x_t. This confirms Cochrane's (2009) result that Taylor rule parameters are not identified under determinacy via this simple form of OLS learning. Away from this case, there will either be two or three discrete solutions for the model's parameters.

However, we previously argued that sunspots were observable to agents. Hence, agents using the perceived law of motion (PLM) (3.1) are not using all available information. If they instead run the regression:

$$x_{t+1} = a_1 x_t + a_2 x_{t-1} + a_3 x_{t-2} + b\mathbb{E}^*_{t-1} x_t + c + d_1' \zeta_t + d_0' \zeta_{t+1} + m_\varepsilon \varepsilon_{t+1}, \\ \varepsilon_{t+1} \sim \mathrm{NIID}(0,1), \tag{3.2}$$

[8] The equations also have a unique solution when either $\alpha = 0$ and $\rho = \frac{1}{\beta}$, or when $\rho = 0$. However, these two cases are observationally equivalent.

then all parameters will apparently be identified, providing $d_0 \neq 0$. For example, in the case where $\dim \zeta_t = 1$ we have: $\frac{1}{\beta} = a_1 + \frac{d_1}{d_0}$ and $\rho = b - \frac{d_1}{d_0}$. We also have the over-identifying restriction $a_3 + \left(a_1 + \frac{d_1}{d_0}\right)\left(b - \frac{d_1}{d_0}\right)^2 = -a_2\left(b - \frac{d_1}{d_0}\right)$. When $\dim \zeta_t > 1$, these equalities must hold for each non-zero component of d_0 and the corresponding component of d_1, giving further over-identifying restrictions. Unfortunately, since the estimated value of d_0 will be non-zero with probability one (even under a MSV solution with $m_\zeta = 0$), under (3.2) although it may seem like we have identified a non-MSV solution, we must continue to place positive probability on being in a MSV solution, so the identification here is illusory. Furthermore, agents generally have no grounds for believing that $m_{\varepsilon,t}$ and $m_{\zeta,t}$ are indeed constant. This means that the standard errors on their parameter estimates should be bounded away from zero even asymptotically, further dashing any hope of identification.

3.3. Identification by learning from learners

Although agents cannot identify structural parameters via running either of the regressions given in the last section, if one sophisticated agent realises that everyone else is running these regressions in order to form expectations then that sophisticated agent will be able to identify parameters.

Since we did not use the rationality of expectations in deriving equation (2.4), it must always be the case that:

$$x_t = (\alpha + \rho)x_{t-1} - \alpha\rho x_{t-2} + \beta \mathbb{E}_t^* x_{t+1} - \beta\rho \mathbb{E}_{t-1}^* x_t + (1 - \rho)\gamma + \sigma\varepsilon_t. \tag{3.3}$$

The only thing stopping us from running a regression of this form in order to identify β is the endogeneity of $\mathbb{E}_t^* x_{t+1}$. But if agents are forming expectations using (3.1) or (3.2) then we know that $d_{1,t-1}' \zeta_t$ is a valid instrument for $\mathbb{E}_t^* x_{t+1}$ (where $d_{1,t-1}$ is the estimated values of d_1 using information up to period $t - 1$ at the latest)[9], since ζ_t is uncorrelated with ε_t by assumption. Hence, one potential way of achieving identification would be to run a standard IV-regression. However, this is unlikely to be very efficient as it discards a lot of information.

We can do considerably better here by considering the structure of the implied actual law of motion (ALM). Note that if everyone is forming expectations by running the regression (3.1) or (3.2), then:

$$\begin{aligned} x_t = \left(1 - \beta a_{1,t-1}\right)^{-1} \big[&(\alpha + \rho + \beta a_{2,t-1})x_{t-1} + \left(\beta a_{3,t-1} - \alpha\rho\right)x_{t-2} + \beta(b_{t-1} - \rho)\mathbb{E}_{t-1}^* x_t \\ &+ [(1 - \rho)\gamma + \beta c_{t-1}] + \beta d_{1,t-1}' \zeta_t + \sigma\varepsilon_t \big], \end{aligned}$$

where time subscripts on the regression coefficients again refer to agents' estimates using information up to period $t - 1$ at the latest. We do not specify at this point if these estimates are the result of recursive least squares (RLS—equivalent to OLS), constant gain least squares (CGLS), or some other estimation method. In the appendix, section 7.2 we analyse e-stability, which will

[9] We are assuming that the OLS agents adopt the standard convention of forming expectations using parameter estimates from previous periods' observations. When they are allowed to use current observations then we can proxy the estimates with current observations by the estimates with lagged ones to avoid further endogeneity issues.

determine convergence of the naïve agents' beliefs under RLS; but this will not be important for the analysis of the convergence of the beliefs of our one sophisticated agent.

Using the ALM above, we can estimate the model's structural parameters by conditional maximum likelihood (ML). The conditional log-likelihood is given by:

$$\begin{aligned}
&\log f(x_1,\dots,x_T|x_0,x_{-1},\mathbb{E}_0^* x_1,\zeta_1,\dots,\zeta_T,h_0,\theta)\\
&=\sum_{t=1}^{T}\log f(x_t|x_{t-1},x_{t-2},\mathbb{E}_{t-1}^* x_t,\zeta_t,h_0,\dots,h_{t-1},\theta)\\
&=-\frac{T}{2}\log 2\pi\ +\sum_{t=1}^{T}\left[\log\left|1-\beta a_{1,t-1}\right|-\log\sigma-\frac{1}{2\sigma^2}(x_t-\mu_t)^2\right]
\end{aligned}$$

where $h_t=[a_{1,t}\quad a_{2,t}\quad a_{3,t}\quad b_t\quad c_t\quad d_{1,t}']'$, $\theta=[\alpha\quad\beta\quad\gamma\quad\rho\quad\sigma]'$,

$$\mu_t\coloneqq(\alpha+\rho)x_{t-1}-\alpha\rho x_{t-2}+\beta\mathbb{E}_t^* x_{t+1}-\beta\rho\mathbb{E}_{t-1}^* x_t+(1-\rho)\gamma,$$

and:

$$\mathbb{E}_t^* x_{t+1}=a_{1,t-1}x_t+a_{2,t-1}x_{t-1}+a_{3,t-1}x_{t-2}+b_{t-1}\mathbb{E}_{t-1}^* x_t+c_{t-1}+d_{1,t-1}'\zeta_t. \tag{3.4}$$

Note that in introducing the conditioning on $h_0,\dots,h_{t-1}$ in the first equality we have used the fact that $h_0,\dots,h_{t-1}$ are deterministic functions of $x_{-1},\dots,x_{t-1}$.

The first order conditions then imply that[10]:

$$\begin{gathered}
0=\sum_{t=1}^{T}(x_{t-1}-\hat\rho x_{t-2})(x_t-\hat\mu_t)\\
0=\sum_{t=1}^{T}\left[\mathbb{E}_t^* x_{t+1}(x_t-\hat\mu_t)-\frac{a_{1,t-1}\hat\sigma^2}{1-\hat\beta a_{1,t-1}}\right]\\
0=\sum_{t=1}^{T}\left(x_{t-1}-\hat\alpha x_{t-2}-\hat\beta\mathbb{E}_{t-1}^* x_t-\gamma\right)(x_t-\hat\mu_t)\\
0=\sum_{t=1}^{T}(x_t-\hat\mu_t),\qquad \hat\sigma^2=\frac{1}{T}\sum_{t=1}^{T}(x_t-\hat\mu_t)^2
\end{gathered} \tag{3.5}$$

Since the second equation is a polynomial of at least order T in β, in general these equations will have to be solved numerically. However, providing parameters are indeed identified, the resulting estimates will have all the usual desirable properties of ML estimates (consistency, efficiency, asymptotic normality).

To show that the ML estimator does indeed identify parameters, we give an alternative estimator that we are able to prove to be consistent. Since the existence of a consistent estimator implies identification (Gabrielsen 1978), this is sufficient for the consistency and asymptotic normality of

[10] As usual, hats denote estimates.

the ML estimator. This alternative estimator will also have a recursive form, making it convenient for the case in which everyone realises everyone else is learning.

Let $\theta \coloneqq [\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5]' = [(1-\rho)\gamma \quad \alpha+\rho \quad -\alpha\rho \quad \beta \quad -\beta\rho]'$ be a vector of parameters to be estimated, and let:

$$z_t \coloneqq \begin{bmatrix} 1 & x_{t-1} & x_{t-2} & \begin{pmatrix} a_{2,t-1}x_{t-1} + a_{3,t-1}x_{t-2} + \\ b_{t-1}\mathbb{E}^*_{t-1}x_t + c_{t-1} + d'_{1,t-1}\zeta_t \end{pmatrix} & \mathbb{E}^*_{t-1}x_t \end{bmatrix}'. \tag{3.6}$$

Suppose for the moment that an oracle told us the value of β. Then by running the regression:

$$\left(1-\beta a_{1,t-1}\right)x_t = z_t'\theta + \sigma\varepsilon_t, \qquad \varepsilon_t \sim \mathrm{NIID}(0,1), \tag{3.7}$$

we could identify all parameters, even if we forgot what the oracle had told us as soon as the regression had been run. In particular $\hat{\sigma}$ is the standard deviation of the shock, $\hat{\beta} = \hat{\theta}_4$, $\hat{\rho} = -\frac{\hat{\theta}_5}{\hat{\beta}} = -\frac{\hat{\theta}_5}{\hat{\theta}_4}$, $\hat{\gamma} = \frac{\hat{\theta}_1}{1-\hat{\rho}} = \frac{\hat{\theta}_1\hat{\theta}_4}{\hat{\theta}_4+\hat{\theta}_5}$, and $\hat{\alpha}$ is given by either $\hat{\theta}_2 - \hat{\rho} = \frac{\hat{\theta}_2\hat{\theta}_4+\hat{\theta}_5}{\hat{\theta}_4}$ or $-\frac{\hat{\theta}_3}{\hat{\rho}} = \frac{\hat{\theta}_3\hat{\theta}_4}{\hat{\theta}_5}$. (The two estimates of α may be near-optimally combined to give $\hat{\alpha} = \frac{\hat{\theta}_3\hat{\theta}_4\hat{\theta}_5 s_{\theta,22} + \hat{\theta}_4(\hat{\theta}_2\hat{\theta}_4+\hat{\theta}_5)s_{\theta,33} - (\hat{\theta}_2\hat{\theta}_4\hat{\theta}_5+\hat{\theta}_3\hat{\theta}_4^2+\hat{\theta}_5^2)s_{\theta,23}}{\hat{\theta}_5^2 s_{\theta,22} + \hat{\theta}_4^2 s_{\theta,33} - 2\hat{\theta}_4\hat{\theta}_5 s_{\theta,23}}$, where $\begin{bmatrix} s_{\theta,22} & s_{\theta,23} \\ s_{\theta,32} & s_{\theta,33} \end{bmatrix}$ is the estimated covariance matrix of $\begin{bmatrix} \hat{\theta}_2 \\ \hat{\theta}_3 \end{bmatrix}$.)

Now let $Z_T \coloneqq \begin{bmatrix} z_1' \\ \vdots \\ z_T' \end{bmatrix}$, $x \coloneqq \begin{bmatrix} x_1 \\ \vdots \\ x_T \end{bmatrix}$ and $y \coloneqq \begin{bmatrix} a_0 x_1 \\ \vdots \\ a_{T-1}x_T \end{bmatrix}$. Then the (OLS) estimated value of θ is given by:

$$\hat{\theta} = (Z_T'Z_T)^{-1}Z_T'(x - y\beta).$$

To show consistency of this estimator, let us begin by defining a vector of "pseudo-instruments" (variables that we would like to use in place of z_t, were they observable):

$$d_t \coloneqq \begin{bmatrix} 1 & \dfrac{\sigma\varepsilon_{t-1}}{1-\beta a_{1,t-2}} & \dfrac{\sigma\varepsilon_{t-2}}{1-\beta a_{1,t-3}} & d'_{1,t-1}\zeta_t & \dfrac{d'_{1,t-2}\zeta_{t-1}}{1-\beta a_{1,t-2}} \end{bmatrix}'.$$

Denote by $\mathbb{E}^+V$ the unconditional expectation of V that would have obtained were $a_{1,t}$, $a_{2,t}$, $a_{3,t}$, b_t, c_t and $d_{1,t}$ non-stochastic for all t. Then if $J_t \coloneqq \mathbb{E}^+ d_t d_t'$,

$$J_t = \operatorname{diag}\begin{bmatrix} 1 & \dfrac{\sigma^2}{\left(1-\beta a_{1,t-2}\right)^2} & \dfrac{\sigma^2}{\left(1-\beta a_{1,t-3}\right)^2} & d'_{1,t-1}d_{1,t-1} & \dfrac{d'_{1,t-2}d_{1,t-2}}{\left(1-\beta a_{1,t-2}\right)^2} \end{bmatrix},$$ [11]

and if $K_t \coloneqq (\mathbb{E}^+ d_t d_t')^{-1}\mathbb{E}^+ d_t z_t'$,

$$K_t = \begin{bmatrix} 1 & ? & ? & ? & ? \\ 0 & 1 & 0 & a_{2,t-1} + b_{t-1}a_{1,t-2} & a_{1,t-2} \\ 0 & q_{t-2} & 1 & \left(a_{2,t-1} + b_{t-1}a_{1,t-2}\right)q_{t-2} + a_{3,t-1} + b_{t-1}a_{2,t-2} & a_{2,t-2} + a_{1,t-2}q_{t-2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \beta & 0 & \beta a_{2,t-1} + b_{t-1} & 1 \end{bmatrix},$$

[11] The diag operator maps vectors to diagonal matrices with a diagonal with the same elements as the vector, and maps matrices to a vector with the same elements as their diagonal.

where $q_{t-2} = \frac{\alpha+\rho+\beta a_{2,t-2}+\beta(b_{t-2}-\rho)a_{1,t-3}}{1-\beta a_{1,t-2}}$, and ? denotes a term omitted for the sake of space. We also define $\tilde{J}_T \coloneqq \sum_{t=1}^{T} J_t$, and $\widetilde{K}_T \coloneqq \tilde{J}_T^{-1} \sum_{t=1}^{T} J_t K_t$, so if $D \coloneqq \begin{bmatrix} d_1' \\ \vdots \\ d_T' \end{bmatrix}$, $\tilde{J}_T = \mathbb{E}^+ D'D$ and $\widetilde{K}_T = (\mathbb{E}^+ D'D)^{-1} \mathbb{E}^+ D' Z_T$. These definitions are valid as $\tilde{J}_T$ is diagonal, with a strictly positive diagonal, for all t. (Though the elements of the diagonal may tend to 0 asymptotically.) A sufficient condition for the invertability of both K_T and $\widetilde{K}_T$, for all T, is that $\beta \neq 1$, in which case the eigenvalues of K_t and $\widetilde{K}_T$ must be bounded away from 0 asymptotically.

If we go on to define:

$$U_T \coloneqq Z_T - D(D'D)^{-1} D' Z_T,$$

then $D'U = 0$ and:

$$Z_T' Z_T = Z_T' D (D'D)^{-1} D'D (D'D)^{-1} D' Z_T + U_T' U_T.$$

If it were valid to drop the $\mathbb{E}^+$ operators from our expressions for $\tilde{J}_T$ and $\widetilde{K}_T$, asymptotically, then we would have:

$$\Pr\left(\lim_{T\to\infty} \left(\widetilde{K}_T' \tilde{J}_T \widetilde{K}_T + U_T' U_T - Z_T' Z_t\right) = 0\right) = 1. \tag{3.8}$$

Dropping the $\mathbb{E}^+$ operators in this way might be valid, for example, if agents were learning a sunspot solution via RLS, and eventually the dependence between their estimates was sufficiently weak that $a_{1,t}$, $a_{2,t}$, etc. were "near exogenous", in some loose sense. However, rather than making such specific assumptions, we will instead just assume the validity of (3.8), since (3.8) encompasses many other cases, including ones in which $\operatorname*{plim}_{T\to\infty} \widetilde{K}_T$ does not even exist, as it will not under constant gain learning.

Given (3.8), by applying Theorem 1 of Lai and Wei (1982) to the regression (3.7), providing:

1) there exists $\delta > 0$ such that $\limsup_{t\to\infty} \frac{\max\{1, a_{1,t}^2\}}{t^{1-\delta} \min\{1, d_{1,t}' d_{1,t}\}} < \infty$,[12] and
2) there exists $\delta' \geq 0$ such that $\limsup_{t\to\infty} \frac{z_t' z_t}{t^{\delta'}} < \infty$,[13]

then $\hat{\theta} \xrightarrow{a.s.} \theta$. Note that 2) already covers all sub-exponential explosion in $z_t' z_t$. We do not as yet have a proof of consistency for the case with an exponential (or super-exponential) explosion, but our simulation results below certainly suggest that β can still be consistently estimated in this case (though obviously γ cannot be).

Furthermore, under slightly stronger assumptions $\tilde{J}_T^{1/2} \widetilde{K}_T \hat{\theta}$ will be asymptotically normally distributed, implying that we have $\frac{1}{\sqrt{\log T}}$ convergence in the worst case.

[12] Sufficient as $\sum_{t=1}^{\infty} t^{-(1-\delta)} = \infty$ for all $\delta \geq 0$.

[13] Sufficient as $\lim_{T\to\infty} \frac{\sum_{t=1}^{T} t^{\delta'}}{T^{1+\delta'}} < \infty$, $\lim_{T\to\infty} \frac{\log T}{\sum_{t=1}^{T} t^{-(1-\delta)}} = 0$ for all $\delta > 0$, and since $\operatorname{tr} z_t z_t'$ is guaranteed to be between the largest eigenvalue of $z_t z_t'$ and 5 times this quantity.

It is easy to see that these sufficient conditions will hold under any non-exponentially-explosive learning algorithm, with slower than $\frac{1}{\sqrt{T}}$ convergence, such as constant gain least squares, or stochastic gradient learning. Under recursive least squares, there exists $\delta \geq 0$ such that $t^{\frac{1-\delta}{2}} d_{1,t}$ converges in distribution to a normal, (Marcet and Sargent 1992), with $\delta = 0$ only if the real parts of the eigenvalues of the "T" matrix are all less than ${}^{1}/_{2}$.[14] When $\delta > 0$ here, our sufficient conditions will be satisfied, but in the other case, Theorem 1 of Lai and Wei (1982) no longer applies. From their reasoning, we do however have that $\limsup_{t\to\infty} (\hat{\theta} - \theta)'(\hat{\theta} - \theta) < \infty$,
even here, so at worst, beyond a certain point in time standard errors on θ would cease improving. Additionally, we note that a sufficient condition for consistency in this case is that:

$$\limsup_{T\to\infty} \left\| \tilde{J}_T^{-1/2} \tilde{K}_T'^{-1} U_T' U_T \tilde{K}_T^{-1} \tilde{J}_T^{-1/2} \right\| < \infty, \tag{3.9}$$

by Theorem 3 of Lai and Wei (1982). This will hold, for example, if $\alpha = \rho = 0$, so it may be thought of as an additional weak-dependency condition.

We have demonstrated then a range of conditions under which $\hat{\theta}$ is a consistent estimator of θ, in our oracle-aided regression, equation (3.7). Now suppose there is no oracle, but we have received infinitely many periods of data. If we guessed a value for β, we could repeat the "oracle" exercise with the guessed value and we would end up with an alternative estimate for β (namely $\hat{\theta}_3$). We can thus think of this as a fixed-point problem. In general our guess of β and the estimated value will not coincide, but we know that they must coincide at least once, namely when our guess is the true value. Thus if the (infinite-data) fixed-point problem has a unique solution for β, then we know that value must be the true value. Hence, if in finite samples this fixed-point problem also has a unique solution, that solution must be a consistent estimator of β, at least when the conditions discussed above hold.

We proceed to establish the uniqueness of the solution to the fixed-point problem, by establishing a closed form solution. Let $e_4 \coloneqq [0 \quad 0 \quad 0 \quad 1 \quad 0]'$. Then the fixed-point problem may be expressed as finding the value of $\hat{\beta}$ for which:

$$\hat{\beta} = e_4'(Z'Z)^{-1}Z'(x - y\hat{\beta}).$$

Consequently:

$$\hat{\beta} = \frac{e_4'(Z'Z)^{-1}Z'x}{1 + e_4'(Z'Z)^{-1}Z'y}.$$

Armed with a consistent estimator of $\hat{\beta}$, all other parameters may be estimated consistently by following our oracle procedure. In particular, the consistent estimator of θ is:

$$\begin{aligned} \hat{\theta}^{\mathrm{2SLS}} &= (Z'Z)^{-1}Z' \left[x - y \frac{e_4'(Z'Z)^{-1}Z'x}{1 + e_4'(Z'Z)^{-1}Z'y} \right] \\ &= (I + (Z'Z)^{-1}Z'ye_4')^{-1}(Z'Z)^{-1}Z'x \end{aligned} \tag{3.10}$$

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[14] These eigenvalues are given in the appendix, 7.2.

$$= (Z'Z + Z'ye_4')^{-1}Z'x,$$

which turns out to be equal to the 2SLS-IV estimator when $\left(a_{2,t-1}x_{t-1} + a_{3,t-1}x_{t-2} + b_{t-1}\mathbb{E}_{t-1}^{*}x_t + c_{t-1} + d_{1,t-1}'\zeta_t\right)$ is used as an instrument for $\mathbb{E}_t^{*}x_{t+1}$.

This gives us the following proposition:

Proposition 1: *Suppose the economy is made up of agents that are all forming expectations through running regressions of the form of* (3.1) *or* (3.2)*, with* $\dim \zeta_t > 0$*. Let* $\hat{\theta}^{\mathrm{2SLS}}$ *be the estimator defined by equation* (3.10)*, and suppose that:*

1) *the weak-dependence condition* (3.8) *holds,*
2) *there exists* $\delta > 0$ *such that* $\limsup_{t\to\infty} \frac{max\{1,a_{1,t}^2\}}{t^{1-\delta}\, min\{1,d_{1,t}'d_{1,t}\}} < \infty$*, and*
3) *there exists* $\delta' \geq 0$ *such that* $\limsup_{t\to\infty} \frac{z_t'z_t}{t^{\delta'}} < \infty$*,*

Then if one of the following conditions holds:

a) *the agents learn by any algorithm with slower than* ${}^{1}\!/_{\sqrt{t}}$ *convergence, such as constant gain least squares, stochastic gradient learning, or recursive least squares in the case in which the eigenvalues of the "*T*" matrix (defined in appendix 7.2) are greater than* ${}^{1}\!/_{2}$*,*
b) *the agents learn a sunspot solution,*
c) *the agents learn by recursive least squares, or another algorithm under which* $\sqrt{t}d_{1,t}'$ *converges in distribution, and the second weak-dependence condition* (3.9) *holds,*

then the 2SLS-like estimator $\hat{\theta}^{\mathrm{2SLS}}$ *is consistent.*

Since the existence of a consistent estimator implies parameter identification under maximum likelihood, we have the following immediate corollary:

Corollary 1.1: *Under the conditions of* ***Proposition 1****, the maximum likelihood estimator given by the solution to the FOCs,* (3.5) *is consistent.*

Note that the consistency of these estimators is in spite of the convergence of $a_{1,t}$, $a_{2,t}$, etc. rather than because of this convergence. Indeed, the worse the learning process that is determining $a_{1,t}$, $a_{2,t}$, etc., the faster this more sophisticated agent will learn the structural parameters of the model. So for example, if almost all agents are using stochastic gradient learning or constant gain least squares, then learning structural parameters is likely to be particularly easy. Likewise if $a_{1,t}$, $a_{2,t}$, etc. never converge then learning the structural parameters is again likely to be fast. This result is related to Cochrane's (2009) claim that with unsophisticated learning it is only in the explosive case that structural parameters may be identified, but here we have identification quite generally.

3.4. Learning from MSV learners

It is natural to wonder the extent to which our results are driven by the fact that the agents in the economy are learning and forming expectations using equation (3.1) or (3.2), rather than the more traditional MSV form:

$$x_{t+1} = a_1x_t + a_2x_{t-1} + c + m_\varepsilon\varepsilon_{t+1}, \qquad \varepsilon_{t+1}\sim\mathrm{NIID}(0,1). \tag{3.11}$$

Since many REE do not have a representation in this form, by estimating (3.11) the agents in the economy are already putting a prior probability of zero on any non-fundamental solution, which is certainly not justified in the absence of transversality constraints limiting x_t to asymptotic stationarity. Nonetheless, even given these priors, when agents observe a stationary realisation of x_t they will still not be able to work out the value of β, as there are observationally equivalent MSV solutions. So, it remains an interesting question whether or not β can be identified from examining these learners.

The argument of the previous section would suggest using $a_{2,t-1}x_{t-1} + c_{t-1}$ as an instrument for $\mathbb{E}_t^* x_{t+1}$. Proving the general validity of this instrument in the MSV set-up is tricky, however. This is clearest when $\alpha = \rho = 0$, in which case, asymptotically $x_{t+1} = m_\varepsilon \varepsilon_{t+1}$, if parameters converge. With no serial correlation in x_t, finding "pseudo-instruments" (i.e. potential elements of d_t) that are correlated with $\mathbb{E}_t^* x_{t+1}$ and $\mathbb{E}_{t-1}^* x_t$, but not with ε_{t-1} or ε_{t-2} is non-trivial.

Suppose that $\frac{1}{\kappa_a(t)}\begin{bmatrix} a_{1,t} - a_{1,\infty} \\ a_{2,t} - a_{2,\infty} \end{bmatrix}$ tends in distribution to some non-degenerate distribution, as $t \to \infty$, for some function $\kappa_a(t)$, and some constants $a_{1,\infty}$ and $a_{2,\infty}$. Then under any "reasonable" estimator (including the RLS, CGLS etc. estimators):

$$\begin{gathered} \liminf_{t\to\infty} \kappa_a(t)^2 \operatorname{cov}\big(a_{1,t}, \varepsilon_t \varepsilon_{t-1}\big) > 0, \\ \liminf_{t\to\infty} \kappa_a(t)^2 \operatorname{cov}\big(a_{2,t}, \varepsilon_t \varepsilon_{t-2}\big) > 0, \& \\ \limsup_{t\to\infty} \kappa_a(t)^2 \operatorname{cov}\big(a_{2,t}, \varepsilon_t \varepsilon_{t-1}\big) = 0. \end{gathered} \tag{3.12}$$

Thus if we define:

$$d_t \coloneqq \begin{bmatrix} 1 & \dfrac{\sigma\varepsilon_{t-1}}{1-\beta a_{1,t-2}} & \dfrac{\sigma\varepsilon_{t-2}}{1-\beta a_{1,t-3}} & \dfrac{\sigma\kappa_a(t)^2\varepsilon_{t-1}^2\varepsilon_{t-3}}{1-\beta a_{1,t-2}} & \dfrac{\sigma\kappa_a(t)^2\varepsilon_{t-1}\varepsilon_{t-2}\varepsilon_{t-4}}{1-\beta a_{1,t-2}} \end{bmatrix}', \tag{3.13}$$

then providing $\liminf_{t\to\infty} t^{1-\delta}\kappa_a(t)^2 > 0$ for some $\delta > 0$, the previous proof goes through.[15] Of course, under recursive least squares learning $\kappa_a(t) = \frac{1}{\sqrt{t}}$ when the eigenvalues of the "T" matrix are less than $1/2$, so this sufficient condition does not hold. While the second weak-dependence condition (3.9) could be generalised to this case, it seems highly implausible that it would hold here, due to the convoluted nature of our "pseudo-instruments".[16]

The convoluted nature of these pseudo-instruments also suggests that our actual-instrument vector, z_t may be a rather poor instrument. One other possibility that could be used as an additional instrument is $a_{1,t-1}$, since it is correlated with the first term of $\mathbb{E}_t^* x_{t+1}$. Indeed, it is easy to see that whether agents are learning from (3.11), or one of our more general laws, (3.1) or (3.2), the asymptotically optimal choice of instruments is:

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[15] We also need to adjust the definition of $\mathbb{E}_t^+$ so that only the $a_{1,t}$ in the denominator of the ALM of x_t is treated as non-stochastic.

[16] Since completing this paper, we discovered the results of Christopeit and Massmann (2010) who were able to prove consistency in an RLS learning of the MSV solution context, for a simple model, using a more direct technique. In future work we intend to investigate whether their proof techniques may be generalised to cover regressions such as these.

$$z_t^* := \begin{bmatrix} z_t \\ a_{1,t-1} z_t \end{bmatrix}$$

since $\mathbb{E}_t x_{t+1} = F z_t^* + a_{1,t-1}\varepsilon_t$ for some non-stochastic, constant matrix F, and this is not true for any proper subset of these instruments. We then have the following generalisation of ***Proposition 1*** and ***Corollary 1.1*** for this choice of instruments:

Proposition 2: *Suppose the economy is made up of agents that are all forming expectations through running regressions of the form of* (3.1)*,* (3.2) *or* (3.11)*. Let* $z_t^* = [z_t' \quad a_{1,t-1} z_t']'$*, where* z_t *is defined by equation* (3.6)*, and let* $Y := Z + y e_4'$*,* $Z^* := [z_1^* \quad \cdots \quad z_T^*]'$*, and:*

$$\hat{\theta}_T^{\mathrm{AEIV}} := \left(Y' Z^* \left(Z^{*\prime} Z^*\right)^{-1} Z^{*\prime} Y\right)^{-1} Z^* \left(Z^{*\prime} Z^*\right)^{-1} Z^{*\prime} x.$$

Then if either:

i) (3.1) *or* (3.2) *is being used, and conditions 1), 2) and 3) of* ***Proposition 1*** *hold, or:*
ii) (3.11) *is being used and:*
   1) the weak-dependence condition (3.8) *holds (with* d_t *defined by* (3.13)*), and,*
   2) there exists $\delta' \geq 0$ *such that* $\limsup_{t\to\infty} \frac{a_{1,t}^2}{t^{\delta'}} < \infty$ *and* $\limsup_{t\to\infty} \frac{z_t' z_t}{t^{\delta'}} < \infty$*,*

and one of the following further conditions holds also:

a) the agents learn by any reasonable[17] *algorithm which converges in distribution, but slower than* ${}^{1}/_{\sqrt{t}}$*, such as stochastic gradient learning, or recursive least squares in the case in which the eigenvalues of the "*T*" matrix (defined in appendix 7.2) are greater than* ${}^{1}/_{2}$*,*

b) the agents learn a sunspot solution,

c) the agents learn by recursive least squares on regression (3.1) *or* (3.2)*, or another algorithm under which* $\sqrt{t} d_{1,t}'$ *converges in distribution,* $\dim \zeta_t > 0$ *and the second weak-dependence condition* (3.9) *holds,*

then the estimator $\hat{\theta}_T^{\mathrm{AEIV}}$ *is consistent and asymptotically efficient.*

Corollary 2.1: *Under the conditions of* ***Proposition 2****, the maximum likelihood estimator given by the solution to the FOCs,* (3.5) *is consistent.*

3.5. Simulation evidence

In light of the slightly obscure nature of some our theoretical conditions, particularly in the recursive least squares (RLS) case, we now present some simulation evidence of the estimator's success in identifying the key β parameter. Figure 1 gives results for economies populated with RLS learners estimating equation (3.2), and Figure 2 gives results for economies populated with RLS learners estimating the MSV form, equation (3.11).

In order to show the estimates performance, for each parameterisation (different rows of the two figures) we generate 2^{14} simulation paths (each of length 2^{8}), and then apply each estimator

[17] Where a reasonable algorithm is defined as one for which (3.12) is satisfied.

considered to each of the resulting paths. In both figures, each of the first three columns corresponds to a different estimator. For both figures, column 1 is our original 2SLS estimator, column 2 is the asymptotically efficient IV one (henceforth, AEIV) and column 3 is the ML estimator[18]. In each graph of the first three columns, we plot the $2.5\%, 5.0\%, 7.5\%, \dots, 97.5\%$ percentiles of the estimator's distribution. For convenience, the quartiles are given in solid rather than dotted lines. The final column of both figures gives the 95% trimmed root mean squared error (RMSE) of the estimators.[19] In this column, the dotted line corresponds to the 2SLS estimator, the dashed to the AEIV one, and the solid to the ML one.

In each simulation run, there was a "burn-in" time of 32 periods during which time expectations were set to their value under the SMSV solution (defined in section 2.2), plus $\sum_{i=1}^{\dim \zeta_t} \zeta_{t,i} + \zeta_t^{\mathrm{B}}$, where ζ_t^{B} is an additional, unobservable, $\mathrm{NIID}(0,1)$ shock. This was done purely in order to help the OLS learners converge, and our estimators were only run on simulated data from the end of the burn-in period. Additionally, the OLS learners' estimates were constrained to have each parameter in $[-1000,1000]$, to prevent numerically unstable hyper-explosions with super-exponential growth. This is in the spirit of the "projection facility" invoked by Marcet and Sargent (1989).

[18] Obtaining a global solution to the numerical maximum likelihood was too slow to permit us to perform as many replications as necessary. Instead then, we start the local maximisation algorithm at the AEIV solution, denoted $\hat{\beta}_t^{\mathrm{AEIV}}$, and constrain the ML estimate of β to be greater than $\max\left[\{0\} \cup \left\{1/a_{1,t-1} \,\middle|\, \hat{\beta}_t^{\mathrm{AEIV}} > 1/a_{1,t-1}, 1 \le t \le T\right\}\right]$ and less than $\min\left[\{0\} \cup \left\{1/a_{1,t-1} \,\middle|\, \hat{\beta}_t^{\mathrm{AEIV}} < 1/a_{1,t-1}, 1 \le t \le T\right\}\right]$.

[19] I.e. the RMSE after first discarding any observations below the 2.5% percentile or above the 97.5% percentile These outliers are trimmed to limit the damage caused by the numerical errors that are introduced by the occasional explosive, or near-explosive, path.

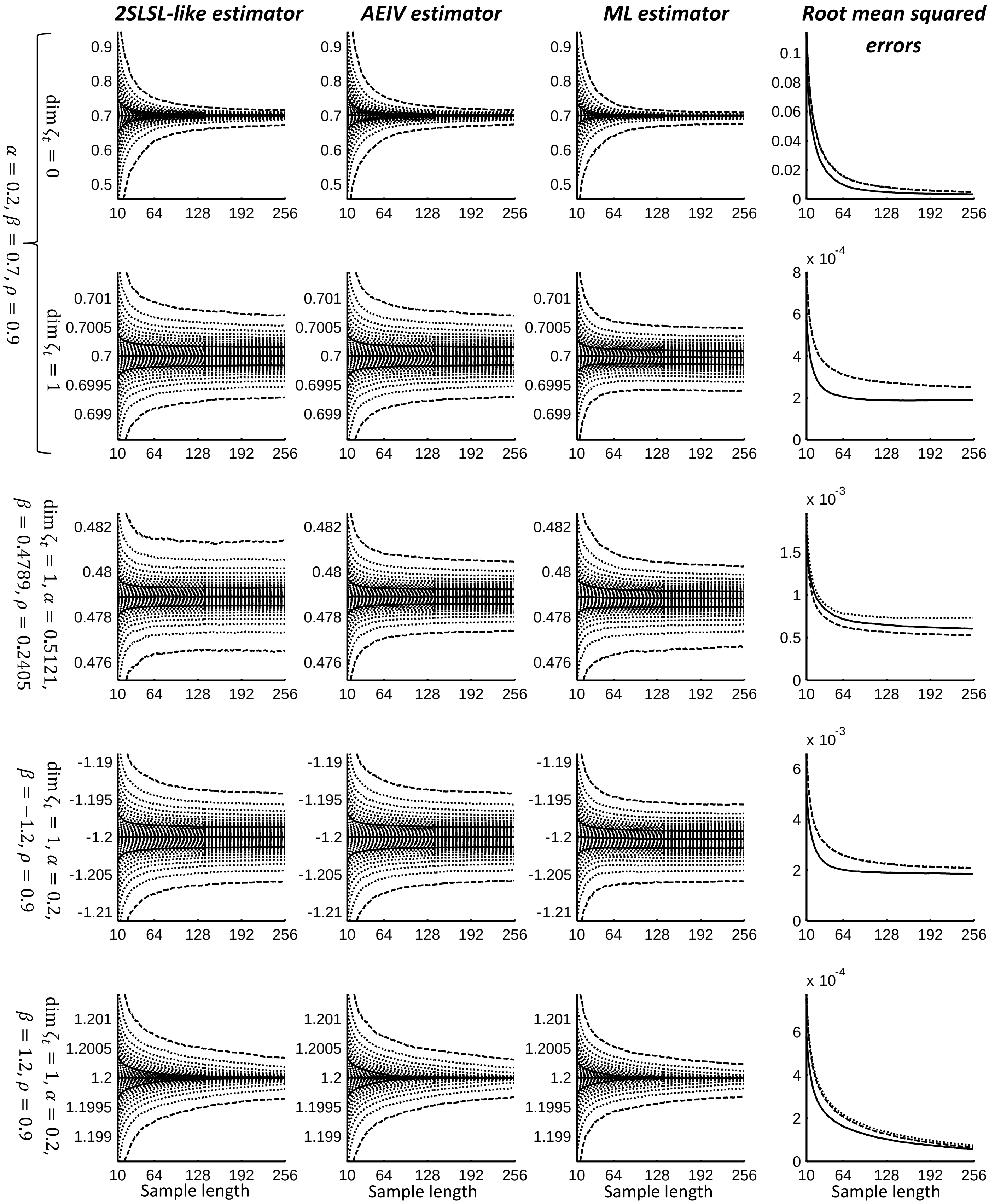

Figure 1: Distribution properties of the estimates of β, from 2^{14} runs, when agents estimate equation (3.2) using OLS.

See text (section 3.5) for full details.

The first two rows of graphs in Figure 1, and the first row in Figure 2, are all generated with $\alpha = 0.2$, $\beta = 0.7$, $\rho = 0.9$, $\sigma = 0.001$ and $\mathbb{E}x_t = 0.005$. These parameters mean there is a unique stationary MSV solution, which is also the only e-stable MSV solution. The graphs in the first row of Figure 1 are with $\dim \zeta_t = 0$, while those in the second have $\dim \zeta_t = 1$. Obviously, in Figure 2 we always set $\dim \zeta_t = 0$. As was expected, the ML estimator dominates the other two, which are practically indistinguishable here. The initial rate of convergence is very quick for all three estimators, but beyond a certain point, convergence certainly seems to slow, in line with our $\frac{1}{\sqrt{\log T}}$ convergence finding. However, although the rate of improvement is slow, the level of the RMSE is low enough that this is unlikely to be a problem in practice.

In the next row of both figures, we repeat the exercise with $\alpha = 0.5121$, $\beta = 0.4789$ and $\rho = 0.2405$. These values were selected as they result in dynamics under full-information that are observationally equivalent to our original ones. Convergence here is slower since two of the eigenvalues of the "T" map are now greater than $1/_2$. There is also clearly large upwards bias in finite samples when agents are estimating (3.11). Surprisingly, it appears the AEIV estimator dominates the ML one in this case, whichever equation is being estimated. Nonetheless, asymptotically our estimators appear to have very similar properties.

In the penultimate row of the figures we show the results when $\alpha = 0.2$, $\beta = -1.2$ and $\rho = 0.9$. This is in the indeterminate region of the parameter space, but still in a region in which the MSV solution is e-stable. Performance appears similar to performance in the $\beta = 0.7$ case.

Finally, in the last row of both figures we show the behaviour of our estimators in an indeterminate region of the parameter space in which the SMSV is not e-stable. (In particular we set $\alpha = 0.2$, $\beta = 1.2$ and $\rho = 0.9$.) The underlying instability of the system makes identification easier for our sophisticated agent, giving us better performance than in any other case, whichever equation is being estimated.

The graphs make clear that even in small samples, when agents are estimating (3.2) all three estimators are approximately unbiased, whatever the true parameters, and whatever the value of $\dim \zeta_t$. Moreover, the estimators are highly peaked around the true value, meaning that the RMSE significantly overstates the median absolute error. Hence, people using these estimators can expect their estimated values to be closer to the truth than is suggested by the standard errors.

4. Learning from sophisticated learners

Having established that our ML and 2SLS-like estimators can successfully identify the structural parameters of the model, we now use these techniques to describe our family of misspecification free learning algorithms. Under these algorithms, each agent in the economy will realise that everyone else is learning at the same time as them, and indeed, they will take advantage of this fact to identify the model's structural parameters. By learning these structural parameters, rather than a reduced form equation, agents will be able to disentangle learning which particular solution to the model is being used from the time variation in reduced form parameters caused by simultaneous learning.

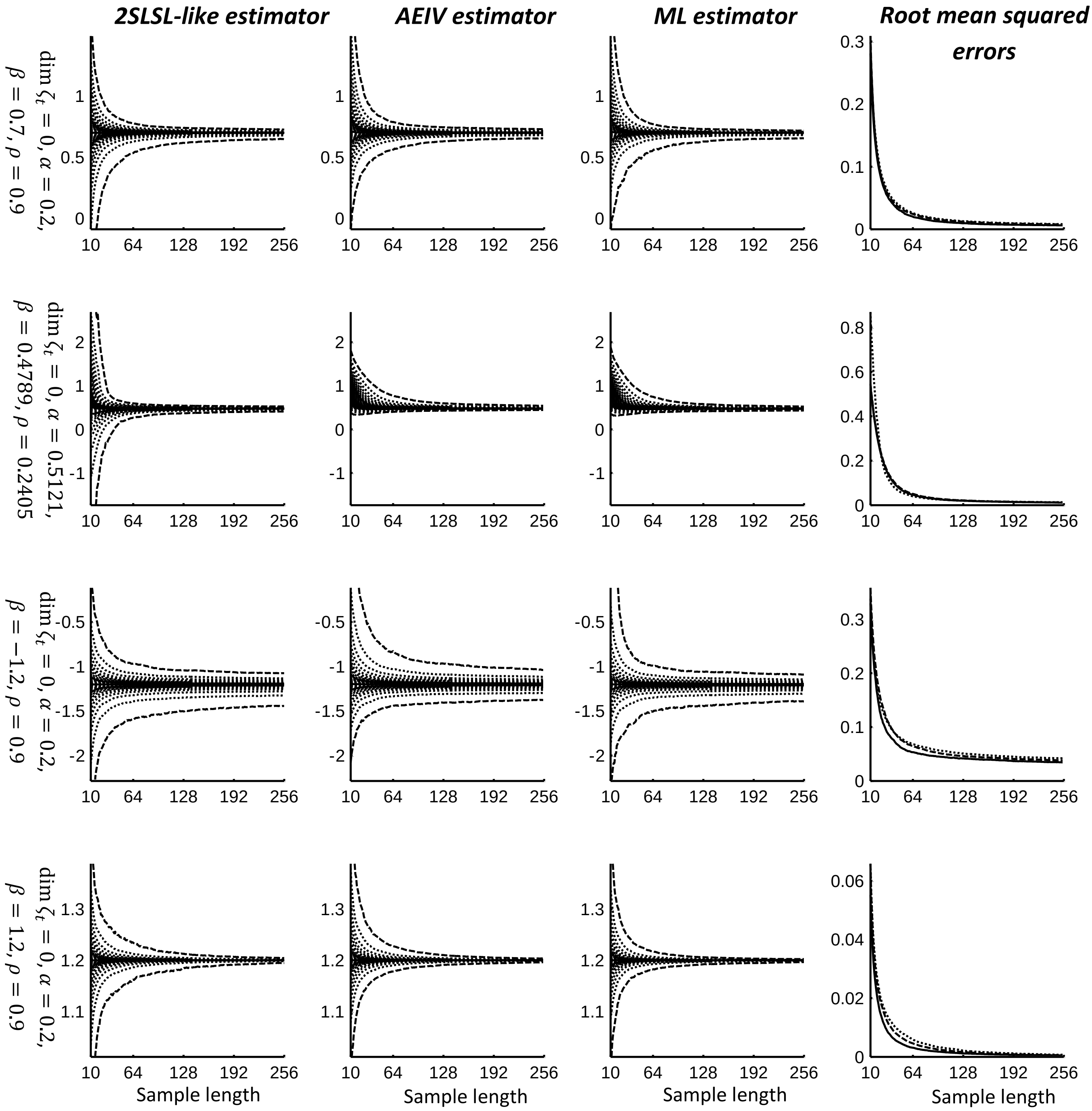

Figure 2: Distribution properties of the estimates of β, from 2^{14} runs, when agents estimate equation (3.11) using OLS.

See text (section 3.5) for full details.

4.1. General results

Suppose for the moment that $m_{\varepsilon,t}$ and $m_{\zeta,t}$ are public knowledge and hence do not have to be estimated, even when no one knows any of the other structural parameters.

Suppose further that everyone is learning using the ML or 2SLS-like estimator from section 3.3. Providing agents continue to use an expression of the form of (3.4) to form expectations, where now $a_{1,t}$ etc. will be functions of estimated structural parameters, this will be valid.

In particular, we might suppose that agents treat their estimate of structural parameters as the true values and set:

$$\begin{aligned} a_{1,t} &= \frac{1}{\hat{\beta}_t}\left(1-\frac{\hat{\sigma}_t}{m_{\varepsilon,t}}\right), \qquad a_{2,t} = -\frac{1}{\hat{\beta}_t}(\hat{\alpha}_t+\hat{\rho}_t), \qquad a_{3,t} = \frac{1}{\hat{\beta}_t}\hat{\alpha}_t\hat{\rho}_t, \\ b_t &= \hat{\rho}_t + \frac{1}{\hat{\beta}_t}\frac{\hat{\sigma}_t}{m_{\varepsilon,t}}, \qquad c_t = -\frac{1}{\hat{\beta}_t}(1-\hat{\rho}_t)\hat{\gamma}_t, \qquad d_{1,t} = \frac{1}{\hat{\beta}_t}\frac{\hat{\sigma}_t}{m_{\varepsilon,t}}m_{\zeta,t}. \end{aligned} \tag{4.1}$$

(4.1) is reasonable since the actual law of motion implied by equations (2.4) and (3.4) is:

$$\begin{aligned} x_{t+1} = \left(1-\beta a_{1,t}\right)^{-1}\Big[&\left(\alpha+\rho+\beta a_{2,t}+\beta(b_t-\rho)a_{1,t-1}\right)x_t + \left(\beta a_{3,t}-\alpha\rho+\beta(b_t-\rho)a_{2,t-1}\right)x_{t-1} \\ &+\beta(b_t-\rho)a_{3,t-1}x_{t-2}+\beta(b_t-\rho)b_{t-1}\mathbb{E}^*_{t-1}x_t+[(1-\rho)\gamma+\beta c_t+\beta(b_t-\rho)c_{t-1}] \\ &+\beta d'_{1,t}\zeta_{t+1}+\beta(b_t-\rho)d'_{1,t-1}\zeta_t+\sigma\varepsilon_{t+1}\Big], \end{aligned}$$

and so when agents use (4.1), if the agents estimates of structural parameters converge in probability to their true values, then $\mathbb{E}_t x_{t+1} - \mathbb{E}^*_t x_{t+1}$ converges in probability to zero.

If agents believe in the SMSV for some reason, then we might suppose they set:

$$\begin{aligned} \hat{\not{\beta}}_t &= \sqrt{\max\{0,1-4\hat{\alpha}_t\hat{\beta}_t\}}, \qquad a_{1,t} = \hat{\rho}_t + \frac{1-\hat{\not{\beta}}_t}{2\hat{\beta}_t}, \qquad a_{2,t} = -\hat{\rho}_t\frac{1-\hat{\not{\beta}}_t}{2\hat{\beta}_t}, \\ a_{3,t} &= 0, \qquad b_t = 0, \qquad c_t = \frac{2(1-\hat{\rho}_t)\hat{\gamma}_t}{1-2\hat{\beta}_t+\hat{\not{\beta}}_t}, \qquad d_{1,t} = 0. \end{aligned} \tag{4.2}$$

If they do this, again as estimates of structural parameters converge in probability to their true values, $\mathbb{E}_t x_{t+1} - \mathbb{E}^*_t x_{t+1}$ will converge in probability to zero.

Furthermore, from ***Proposition 2*** we immediately have the following two corollaries:

Corollary 2.2: *Suppose that $m_{\varepsilon,t}$ and $m_{\zeta,t}$ are in all agent's period t information set, and $m_{\varepsilon,t} \neq 0$ for all t. Then if:*

1) *all agents form expectations using* (3.4) *and* (4.1)*,*
2) *conditions 1), 2) and 3) of **Proposition 1** hold,*
3) *there exists $\delta > 0$ such that $\liminf_{t\to\infty} t^{1-\delta} m_{\zeta,t} > 0$, and,*
4) *agents estimate structural parameters using either the AEIV estimator defined in **Proposition 2**, or the ML estimator given by the solution to the FOCs,* (3.5)*,*

*then all estimates of **structural** parameters will converge in probability to the true values, and agents' expectations will converge in probability to their values under the full information, rational expectations solution.*

Corollary 2.3: *If:*

1) *all agents form expectations using* (3.4) *and* (4.2)*,*
2) *conditions 1) and 2) of **Proposition 2** hold, and,*
3) *agents estimate structural parameters using either the AEIV estimator defined in **Proposition 2**, or the ML estimator given by the solution to the FOCs,* (3.5)*,*

*then all estimates of **reduced form** parameters will converge in probability to the true values, and agents' expectations will converge in probability to their values under the full information, rational expectations, SMSV solution.*

Note that ***Corollary 2.3*** only guarantees convergence of reduced form parameters, not structural ones. This is because if reduced form parameters converge too quickly, ***Proposition 2*** does not apply. Since there are more structural parameters than reduced form ones in the MSV case, it is quite possible for the reduced form parameters to converge without the structural ones converging. Guaranteeing convergence of reduced form parameters is sufficient for expectations to converge to the SMSV solution, however.

To guarantee the existence of a learning algorithm that will learn an arbitrary solution, we need the following supplemental corollary of ***Corollary 2.2***:

Corollary 2.4: *Suppose that agents do not know $m_{\varepsilon,t}$ and $m_{\zeta,t}$, and each agent i forms the estimate $\hat{m}_{\varepsilon,t}(i)$ and $\hat{m}_{\zeta,t}(i)$ (respectively) of these parameters at t. Suppose further that the mechanism they use for learning these parameters means that either:*

1) *there exists some $T \in \mathbb{Z}$ such that for all $t \geq T$, and all agents i and j, $\hat{m}_{\varepsilon,t}(i) = \hat{m}_{\varepsilon,t}(j)$ and $\hat{m}_{\zeta,t}(i) = \hat{m}_{\zeta,t}(j)$, or,*
2) *for all agents i and j $\operatorname*{plim}_{t\to\infty} \frac{\hat{m}_{\varepsilon,t}(i)}{\hat{m}_{\varepsilon,t}(j)} = 1$ and $\operatorname*{plim}_{t\to\infty} \frac{\left(\hat{m}_{\zeta,t}(i)-\hat{m}_{\zeta,t}(j)\right)'\left(\hat{m}_{\zeta,t}(i)-\hat{m}_{\zeta,t}(j)\right)}{\hat{m}_{\zeta,t}(i)'\hat{m}_{\zeta,t}(i)} = 0$,[20]*

*then if $m_{\varepsilon,t}(i) \neq 0$ for all t and i, and conditions 1), 2) and 4) of **Corollary 2.2** are satisfied, then all estimates of **reduced form** parameters will converge in probability to the true values, and agents' expectations will converge in probability to their values under the full information, rational expectations solution. If in addition condition 3) of **Corollary 2.2** is satisfied, then all estimates of **structural** parameters will also converge.*

The proof of the result under condition 1) of this proposition follows from ***Proposition 2***. Under condition 2) the result follows from the fact that condition 2) implies that asymptotically the measurement error induced by treating an idiosyncratic estimate as an aggregate one is dominated by the signal, so the estimates will remain consistent, at least when x_t is non-explosive.

The set of learning mechanisms covered by ***Corollary 2.3*** and ***Corollary 2.4*** includes a very large number of plausible learning mechanisms. In the below, we mention three of particular interest.

4.2. Guaranteed learning of SMSV solutions

Corollary 2.3 guarantees convergence to any SMSV solution, given minimal conditions. Again, since these technical conditions are a little opaque, in Figure 3 we present simulation evidence demonstrating the broad convergence of our algorithm. The rows of Figure 3 correspond to the same rows of Figure 2 (identical parameters were used).

[20] Condition 1) is strictly encompassed by condition 2), but the former will be more useful in practice.

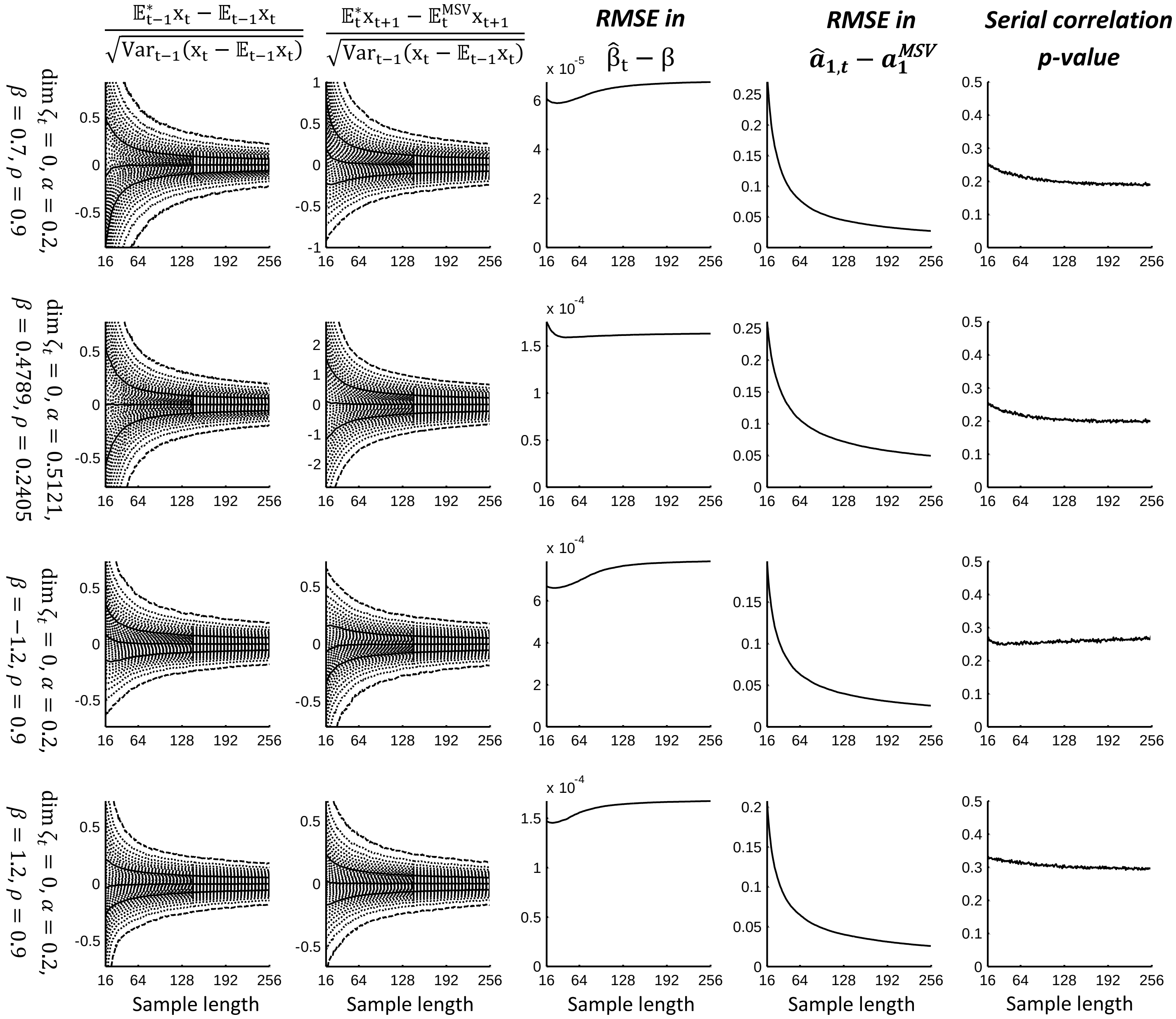

Figure 3: Results from simulations of sophisticated SMSV learners, from 2^{14} runs.
See text (section 4.2) for full details.

As in section 3.5, we make 2^{14} simulation runs, each of length 2^8. For the sake of numerical stability, we again use a projection facility, with all reduced form and structural parameters constrained to lie in the interval $[-1000,1000]$. We also have an eight period burn-in, during which expectations are given by their SMSV solution, plus ζ_t (always a scalar). For all simulations, we use the ML algorithm for parameter estimation, due to its greater efficiency.[21]

[21] Again, we only search for a local maximum, using the constraints as set up in footnote 18. To further increase the chance of finding a global maximum however, each period we try starting the optimisation routine at two different points: last period's estimate, and the AEIV solution.

The first column of Figure 3 presents the distribution of the difference between the expectations formed by our sophisticated agents, and the expectations that would be formed by fully informed, fully rational agents in the same economy, normalised by the full information one-step ahead standard deviation. The second column presents the distribution of the difference between our agents' expectations and the SMSV solution,[22] with the same normalisation. In all cases, it is clear that we have rapid convergence to the SMSV solution, and even faster convergence to rationality.

The third column presents the 95% trimmed RMSE in agents' estimates of β, and the fourth column does the same for a_1^{MSV}. In line with our theoretical results, while β does not appear to converge, agents' estimates of a_1^{MSV} converge to the truth in all cases. (The RMSE in β is nonetheless very small.) Finally, the fifth column presents the mean p-value from a (one-sample) bootstrapped LM test of serial correlation in expectational errors, at one lag. If information is being used fully efficiently, there should be no serial correlation, and these mean p-values should be equal to 0.5. While our found p-values are not quite so high, in all cases they are comfortably above 0.2 at all lags, so an econometrician would not reject the null of no serial correlation, at any standard significance level. Thus although this sophisticated learning algorithm is still not quite fully rational, it is close enough to rationality that users of it could not detect their own deviations from rationality.

Under standard OLS learning, there are non-learnable stationary MSV solutions such as the one in the final row of Figure 3, so by this measure the present learning algorithm is an improvement. However, it is in no sense an answer to Cochrane's (2009) challenge for learnability to "save new-Keynesian models". This learning algorithm is only reasonable if agents already believe that the solution is of the SMSV form, an assumption that is not justified by anything in the model. That dramatically different results may obtain with different learning mechanisms is made clear by the next one presented.

4.3. Learning any sunspot solution (with positive density)

Suppose, that agent i believes that as well as having access to all the same information as them, everyone else in the economy also had access to the additional information that $m_{\varepsilon,t} \equiv m_{\varepsilon,0}$ and $m_{\zeta,t} \equiv m_{\zeta,0}$, where $m_{\varepsilon,0}$ and $m_{\zeta,0}$ are constants, unknown to agent i.

Let us define:

$$e_t := \frac{\hat{\beta}_{t-1}}{\hat{\sigma}_{t-1}}\left[\frac{1}{\hat{\beta}_{t-1}} x_t + \hat{a}_{2,t-1}x_{t-1} + \hat{a}_{3,t-1}x_{t-2} + \hat{\rho}_{t-1}\mathbb{E}_{t-1}^{*}x_t + \hat{c}_{t-1} - \mathbb{E}_t^{*}x_{t+1}\right], \tag{4.3}$$

then:

$$\begin{bmatrix} e_t & \zeta_t' \end{bmatrix}\begin{bmatrix} m_{\varepsilon,t-1} \\ m_{\zeta,t-1} \end{bmatrix} \approx x_t - \mathbb{E}_{t-1}^{*}x_t =: \eta_t^{*},$$

where the approximation is exact when $m_{\varepsilon,t-1} = m_{\varepsilon,0}$. (Away from this point, agent i's estimate of $a_{1,t}$ will differ from the true value, introducing error into their estimates of α_t, etc..)

[22] Given by $\mathbb{E}_t^{\mathrm{MSV}}x_{t+1} = a_1^{\mathrm{MSV}}x_t - a_2^{\mathrm{MSV}}x_{t-1} - c^{\mathrm{MSV}}$

The natural estimate of $m_{\varepsilon,t}$ and $m_{\zeta,t}$ is then:

$$\begin{bmatrix}\widehat{m}_{\varepsilon,t}\\ \widehat{m}_{\zeta,t}\end{bmatrix}=\begin{bmatrix}e_1 & \zeta_1'\\ \vdots & \vdots\\ e_t & \zeta_t'\end{bmatrix}^{+}\begin{bmatrix}\eta_1^*\\ \vdots\\ \eta_t^*\end{bmatrix},$$

where superscript $+$ denotes the Moore-Penrose pseudo-inverse.[23] By the standard properties of least squares estimates, this will converge on the truth, and indeed despite the presence of the approximation in the previous equation this will happen exactly in finite time, providing estimates of other parameters are updated recursively.[24]

In the case we are chiefly concerned with, everyone is learning simultaneously, so by the properties of the Moore-Penrose pseudo-inverse, we will have $\widehat{m}_{\varepsilon,t}\equiv\widehat{m}_{\varepsilon,1}=\frac{e_1}{e_1^2+\zeta_1'\zeta_1}\eta_1^*$ and $\widehat{m}_{\zeta,t}\equiv\widehat{m}_{\zeta,1}=\frac{\zeta_1}{e_1^2+\zeta_1'\zeta_1}\eta_1^*$, ex-post justifying the constancy assumption that motivated the learning method. By varying initial beliefs we may attain any value for η_1^*, and hence any value for $\widehat{m}_{\varepsilon,1}$ and $\widehat{m}_{\zeta,1}$. So with stochastic initial beliefs (a public signal perhaps), any solution is attainable with positive density, and expectations will converge to rationality with probability one (at least given the relevant technical conditions), by ***Corollary 2.4***.[25]

This learning method is readily extended to the case in which agents believe that $m_{\varepsilon,t}$ and $m_{\zeta,t}$ are constant until a certain event occurs. Possible candidates for these events include changes of central bank governors, changes of governments, financial crashes and natural disasters. In this case, each time the event occurs, a new draw for $\widehat{m}_{\varepsilon,t}$ and $\widehat{m}_{\zeta,t}$ will be taken, and they will remain fixed at those values until the event occurs again. In the extreme case in which the event occurs every period, we have that $\widehat{m}_{\varepsilon,t}=\frac{e_t}{e_t^2+\zeta_t'\zeta_t}\eta_t^*$ and $\widehat{m}_{\zeta,t}=\frac{\zeta_t}{e_t^2+\zeta_t'\zeta_t}\eta_t^*$. Since $\mathbb{E}_{t-1}\eta_t^{*2}-\left(\widehat{m}_{\varepsilon,t-1}^2+\widehat{m}_{\zeta,t-1}'\widehat{m}_{\zeta,t-1}\right)\to 0$ as $t\to\infty$, this means $\mathbb{E}_{t-1}\left({\eta_{t+1}^*}^2\right)-\mathbb{E}_{t-1}\left({\eta_t^*}^2\right)\to 0$, so the variance of expectational errors follows a random walk asymptotically, providing endogenous stochastic volatility.

In Figure 4, we show simulations of this learning method, with the exact same set-up as in section 4.2. (We do not bound $\widehat{m}_{\varepsilon,t}$ or $\widehat{m}_{\zeta,t}$ however.) Since initial estimates of e_t are highly inaccurate, we assume all agents update their estimates of $\widehat{m}_{\varepsilon,t}$ and $\widehat{m}_{\zeta,t}$ in each of the first 8 periods after the end of the burn-in (i.e. periods 9 to 16), but not in any future period.

In the two cases in which only the SMSV solution is stationary, expectations asymptotically diverge from rationality. However, there is an initial period of rapid convergence, so it is hard to know if this

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[23] This is of course the standard linear regression formula when $t\ge\dim\zeta_t$.

[24] In this situation, agent i should update their estimates of $a_{1,1}$ in all periods t with $t\ge 1$. I.e. in period t, they should estimate $a_{1,1}$ as $\frac{1}{\beta_1}\left(1-\frac{\widehat{\sigma}_1}{\widehat{m}_{\varepsilon,t}}\right)$. Based on this revised estimate of $a_{1,1}$, they can then re-estimate α_2, etc., and then $a_{1,2}$, etc., and so on. Armed with this set of new estimates, they can then re-estimate $m_{\varepsilon,t}$ and $m_{\zeta,t}$, repeating the entire procedure until they converge on a fixed point. After $1+\dim\zeta_t$ periods have elapsed, there may possible be multiple such fixed points, however, the next period, with probability 1 only one will remain.

[25] The solutions with $\widehat{m}_{\zeta,1}=0$ are not guaranteed to converge, but the set of such solutions is of measure zero in the whole space.

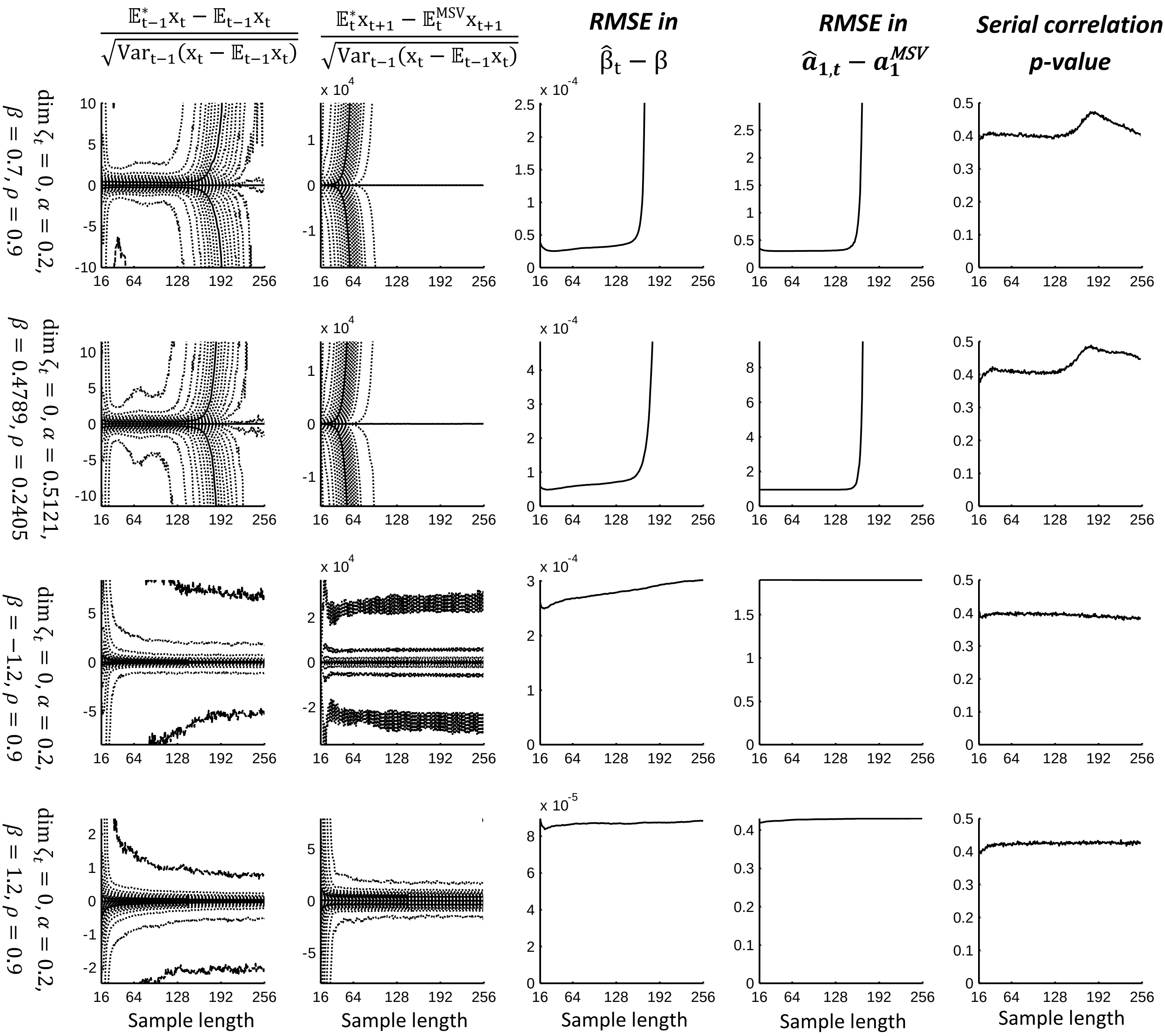

Figure 4: Results from simulations of sophisticated sunspot learners, from 2^{14} runs.
See text (section 4.3) for full details.

divergence is merely driven by the numerical errors stemming from the explosive behaviour of x_t. (Either hypothesis would be consistent with our theoretical results, as these do not cover cases in which x_t grows exponentially or faster.) In the two "indeterminate" cases, expectations rapidly converge to rationality, though not to the MSV solution, implying a sunspot solution has been learnt. While structural parameter estimates are very close to the truth in all cases, they do not appear to be converging. This again is consistent with our theoretical results if reduced form parameters have converged too quickly. Finally, note that there is even less evidence of serial correlation in this sunspot case, so again the agents in the model would not be able to detect their own departure from rationality.

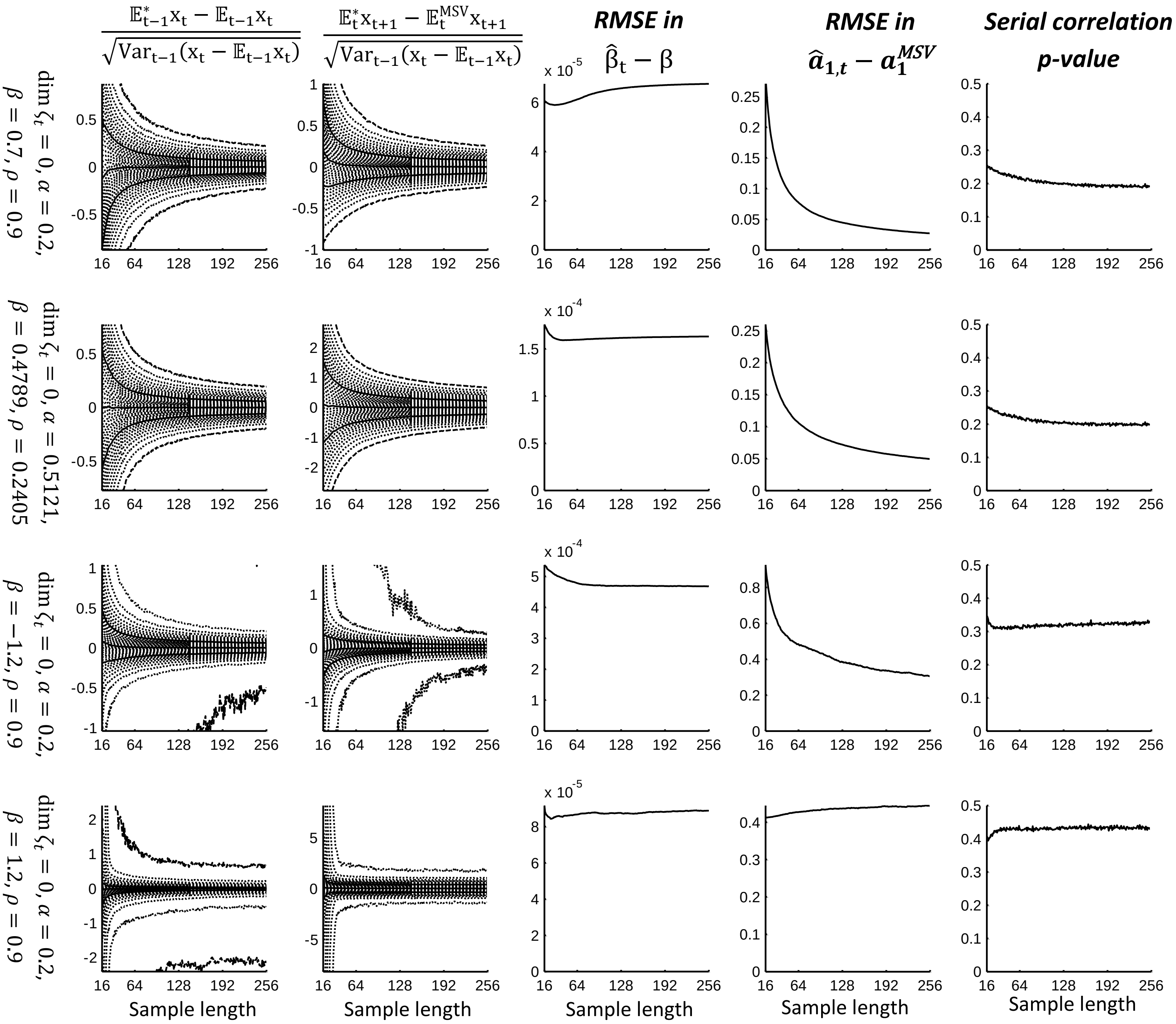

Figure 5: Results from simulations of sophisticated transversality learners, from 2^{14} runs.
See text (section 4.4) for full details.

4.4. Learning in the presence of transversality constraints

Finally, suppose that in the model under consideration, x_t is restricted by a transversality constraint. (To recap, this is not the case for inflation.) Then if agents are ever confident they are in an indeterminate region of the parameter space, they should switch to the SMSV solution. This suggests that agents should begin using the sunspot learning method from the previous section. If however their estimates ever imply that $\left|\hat{\alpha}_t + \hat{\beta}_t\right| < 1$, then they should switch to forming MSV expectations. If at a later date they again come to believe that $\left|\hat{\alpha}_t + \hat{\beta}_t\right| > 1$, they should switch back to the general sunspot solution, with updated values for $\widehat{m}_{\varepsilon,t}$ and $\widehat{m}_{\zeta,t}$.

Figure 5 presents simulations of this learning method. Performance is an amalgam of the previous two cases, with convergence to the SMSV solution under determinacy, and convergence to a sunspot solution otherwise.

5. Conclusion

This paper has set forward a family of macroeconomic learning algorithms that are correctly specified, even along the transition path. Our simulations and theoretical results imply that vastly more equilibria are learnable via these algorithms than via traditional learning methods, implying that learnability cannot be used for equilibrium selection. We have also demonstrated that from observing traditional macroeconomic learners we may identify all a model's structural parameters, providing those traditional learners are running a regression that encompasses the general solution to the model.

The new estimators produced in this paper have many practical applications. In future empirical work we hope to use them to assess whether the Federal Reserve has ever pursued a policy satisfying the Taylor principle, something that was not possible until now due to the non-identification of the key parameter given unobserved, auto-correlated monetary policy shocks. We also hope to look for empirical evidence on whether real world macroeconomic learning is best described by the traditional algorithm or one of our new, misspecification-free methods.

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7. Online appendices

7.1. FREE solutions for arbitrary linear models

We now extend the structure of (2.3) to the general multivariate case:

$$\mathrm{K}x_t = \mathrm{A}x_{t-1} + \mathrm{B}\mathbb{E}_t x_{t+1} + \gamma + \Sigma_s s_t$$

where:

$$s_t = \mathrm{P}s_{t-1} + \Sigma_\varepsilon \varepsilon_t$$

for the arbitrary matrices K, A, B, P, Σ_s and Σ_ε, the vector γ and the shock $\varepsilon_t \sim \mathrm{NIID}(0, I)$. Initially, we suppose that there are no transversality conditions restricting any of the components of x_t.

Again defining the expectational error by $\eta_t \coloneqq x_t - \mathbb{E}_{t-1}x_t$, when B and Σ_s have linearly independent columns, from the properties of the Moore-Penrose pseudoinverse (denoted by $+$), we have that:

$$\begin{aligned}\mathbb{E}_t x_{t+1} = {}& \mathrm{B}^+(\mathrm{K} + \Sigma_s \mathrm{P}\Sigma_s^+ \mathrm{B})x_t - \mathrm{B}^+(\mathrm{A} + \Sigma_s \mathrm{P}\Sigma_s^+ \mathrm{K})x_{t-1} + \mathrm{B}^+\Sigma_s \mathrm{P}\Sigma_s^+ \mathrm{A}x_{t-2} - \mathrm{B}^+\Sigma_s(I - \mathrm{P})\Sigma_s^+\gamma \\ & - \mathrm{B}^+\Sigma_s\Sigma_\varepsilon\varepsilon_t - \mathrm{B}^+\Sigma_s \mathrm{P}\Sigma_s^{-1}\mathrm{B}\eta_t.\end{aligned}$$

As before, without loss of generality we may assume that $\eta_t = M_{\varepsilon,t-1}\varepsilon_t + M_{\zeta,t-1}\zeta_t$, for some sunspot shock ζ_t uncorrelated with ε_t (and satisfying $\mathbb{E}_{t-1}\zeta_t = 0$, $\mathbb{E}_{t-1}\zeta_t\zeta_t' = I$).

Then, if $M_{\varepsilon,t-1}$ has linearly independent columns:

$$\begin{aligned}\mathbb{E}_t x_{t+1} = {}& \mathrm{B}^+\left(\mathrm{K} - \Sigma_s\Sigma_\varepsilon M_{\varepsilon,t-1}^+\right)x_t - \mathrm{B}^+(\mathrm{A} + \Sigma_s \mathrm{P}\Sigma_s^+ \mathrm{K})x_{t-1} + \mathrm{B}^+\Sigma_s \mathrm{P}\Sigma_s^+ \mathrm{A}x_{t-2} \\ & + \mathrm{B}^+\left(\Sigma_s \mathrm{P}\Sigma_s^+ \mathrm{B} + \Sigma_s\Sigma_\varepsilon M_{\varepsilon,t-1}^+\right)\mathbb{E}_{t-1}x_t - \mathrm{B}^+\Sigma_s(I - \mathrm{P})\Sigma_s^+\gamma + \mathrm{B}^+\Sigma_s\Sigma_\varepsilon M_{\varepsilon,t-1}^+ M_{\zeta,t-1}\zeta_t.\end{aligned}$$

This expression no longer contains either ε_t or s_t. Thus, when B and Σ_s have linearly independent columns, almost all rational expectations solutions to the original model are FREE, i.e. they are implementable by agents who cannot observe the model's fundamental shocks.

More generally, there will be transversality conditions restricting some variables, and B and Σ_s will not have linearly independent columns. To solve this case, we closely follow Mavroeidis and Zwols's (2007) presentation of Lubik and Schorfheide's (2003) extension to the irregular case of Sims's (2002) method for solving rational expectations models, which is itself more general than that of Blanchard and Kahn (1980). The majority of the results here that are not due to Mavroeidis, Zwols, Lubik, Schorfheide or Sims were first shown in an earlier working paper by this author (Holden 2008).

With the model set-up as before, let us define $v_t \coloneqq \begin{bmatrix} x_t \\ \mathbb{E}_t x_{t+1} \end{bmatrix}$, $\Gamma_0 \coloneqq \begin{bmatrix} \mathrm{K} & -\mathrm{B} \\ I & 0 \end{bmatrix}$, $\Gamma_1 \coloneqq \begin{bmatrix} \mathrm{A} & 0 \\ 0 & I \end{bmatrix}$, $\mu \coloneqq \begin{bmatrix} \gamma \\ 0 \end{bmatrix}$, $\Psi \coloneqq \begin{bmatrix} \Sigma_s \\ 0 \end{bmatrix}$ and $\Pi \coloneqq \begin{bmatrix} 0 \\ I \end{bmatrix}$. We then have the general canonical form we will solve here:

$$\Gamma_0 v_t = \Gamma_1 v_{t-1} + \mu + \Psi s_t + \Pi\eta_t.$$

In deriving the conditions for the existence of a rational expectations equilibria (REE) below, we will not assume anything about the structure of v_t, η_t, Γ_0, Γ_1, μ, Ψ, Π, P, Σ_s or Σ_ε (beyond the fact that η_t must be chosen subject to $\mathbb{E}_{t-1}\eta_t = 0$). We will also be able to derive sufficient conditions for the existence of a FREE in this fully general case. However, in deriving necessary conditions we will assume that $v_t = \begin{bmatrix} x_t \\ \mathbb{E}_t x_{t+1} \end{bmatrix}$ and $\eta_t = x_t - \mathbb{E}_{t-1} x_t$, as in the above.

By the generalized complex Schur decomposition (also known as the QZ decomposition) (Quarteroni, Sacco, and Saleri 2000) of the matrices Γ_0 and Γ_1, there always exist possibly complex matrices Q, Z, $\Lambda = (\lambda_{i,j})_{i,j}$ and $\Omega = (\omega_{i,j})_{i,j}$ such that $Q^H \Lambda Z^H = \Gamma_0$, $Q^H \Omega Z^H = \Gamma_1$, Q and Z are unitary, Λ and Ω are upper triangular and a superscript H denotes conjugate transpose.

Now let $w_t = Z^H v_t$ for all $t \in \mathbb{Z}$, then if we pre-multiply the canonical form by Q we have:

$$\Lambda w_t = \Omega w_{t-1} + Q(\mu + \Psi s_t + \Pi \eta_t).$$

Providing Γ_0 and Γ_1 do not have zero eigenvalues corresponding to the same eigenvector[26] the QZ decomposition always exists and the set $\left\{ \left| \frac{\omega_{ii}}{\lambda_{ii}} \right| \middle| i \in \{1, \dots, \dim v_t\} \right\} \subseteq \mathbb{R} \cup \{\infty\}$ is unique even though the decomposition itself is not (Sims 2002). Thus, without loss of generality we may assume that for $i < j$, $\left| \frac{\omega_{ii}}{\lambda_{ii}} \right| < \left| \frac{\omega_{jj}}{\lambda_{jj}} \right|$. Let $\bar{u}$ be the number of i for which $\left| \frac{\omega_{ii}}{\lambda_{ii}} \right| \leq 1$ and consider a partition of the matrices under consideration in which in each case the top left block is of dimension $\bar{u} \times \bar{u}$[27].

We may then write:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \begin{bmatrix} Q_{1\cdot} \\ Q_{2\cdot} \end{bmatrix} (\mu + \Psi s_t + \Pi \eta_t). \tag{7.1}$$

The second block of this equation is purely explosive by construction. More generally, we may follow Sims (2002) and allow explosive combinations of variables that do not violate transversality to enter into the upper block. In New-Keynesian models, inflation rates will generally be such a variable.

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[26] This means that there is one or more equation that places no restrictions on either v_t or v_{t-1}. This will create an additional source of indeterminacy in v_t and may also imply that one or more components of ε_t and η_t are linear combinations of the others. We, like both Sims and Lubik & Schorfheide, will not further investigate this avenue.

[27] This means that we are not treating unit roots as explosive. Doing this avoids some minor technical complications.

If agents expect a non-transversality violating path for v_t, from solving forward, following Sims (2002) and Mavroeidis and Zwols (2007), we must have:

$$\begin{aligned} w_{2,t} = \mathbb{E}_t w_{2,t} &= -\mathbb{E}_t \sum_{k=1}^{\infty} (\Omega_{22}^{-1}\Lambda_{22})^{k-1} \Omega_{22}^{-1} Q_{2\cdot}(\mu + \Psi s_{t+k} + \Pi \eta_{t+k}) \\ &= -\sum_{k=0}^{\infty} (\Omega_{22}^{-1}\Lambda_{22})^{k} \Omega_{22}^{-1} Q_{2\cdot}\Psi \mathrm{P}^{1+k} s_t - \left[\sum_{k=0}^{\infty} (\Omega_{22}^{-1}\Lambda_{22})^{k}\right] \Omega_{22}^{-1} Q_{2\cdot}\mu \\ &= S\mathrm{P} s_t + (\Lambda_{22} - \Omega_{22})^{-1} Q_{2\cdot}\mu, \end{aligned}$$

where S is the solution to the Stein equation[28]:

$$\Omega_{22}^{-1}\Lambda_{22} S\mathrm{P} - S = \Omega_{22}^{-1} Q_{2\cdot}\Psi$$

and where the sums are well defined since the eigenvalues of $\Omega_{22}^{-1}\Lambda_{22}$ are strictly in the unit circle by construction (and Ω_{22} is invertible by construction). Note that for S to have linearly independent columns, it is necessary that $\dim w_{2,t} \geq \dim s_t$.

Consequently (following Mavroeidis and Zwols (2007)), $\mathbb{E}_{t+1} w_{2,t} = \mathbb{E}_t w_{2,t}$, and so:

$$\begin{aligned} -\mathbb{E}_{t+1} &\sum_{k=1}^{\infty} (\Omega_{22}^{-1}\Lambda_{22})^{k-1} \Omega_{22}^{-1} Q_{2\cdot}(\mu + \Psi s_{t+k} + \Pi \eta_{t+k}) \\ &= -\mathbb{E}_t \sum_{k=1}^{\infty} (\Omega_{22}^{-1}\Lambda_{22})^{k-1} \Omega_{22}^{-1} Q_{2\cdot}(\mu + \Psi s_{t+k} + \Pi \eta_{t+k}) \end{aligned}$$

i.e.
$$\Omega_{22} S \Sigma_\varepsilon \varepsilon_{t+1} = Q_{2\cdot}\Pi \eta_{t+1} \tag{7.2}$$

(using the fact that Ω_{22} is of full rank and the definition of S). This is the key constraint limiting expectations. If $\mathrm{P} = 0$, then $S = -\Omega_{22}^{-1} Q_{2\cdot}\Psi$ so under the normalisation $\Sigma_\varepsilon = I$, it collapses to the expression given in Lubik and Schorfheide (2003).

By the singular value decomposition (SVD) (Horn and Johnson 1985) of $Q_{2\cdot}\Pi$ and $\Omega_{22} S \Sigma_\varepsilon$ we can write $Q_{2\cdot}\Pi = UDV^H = [U_{\cdot 1} \quad U_{\cdot 2}] \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{\cdot 1}^H \\ V_{\cdot 2}^H \end{bmatrix} = U_{\cdot 1} D_{11} V_{\cdot 1}^H$ and $\Omega_{22} S \Sigma_\varepsilon = \widehat{U}\widehat{D}\widehat{V}^H = [\widehat{U}_{\cdot 1} \quad \widehat{U}_{\cdot 2}] \begin{bmatrix} \widehat{D}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{V}_{\cdot 1}^H \\ \widehat{V}_{\cdot 2}^H \end{bmatrix} = \widehat{U}_{\cdot 1} \widehat{D}_{11} \widehat{V}_{\cdot 1}^H$ where U, V, $\widehat{U}$ and $\widehat{V}$ are unitary and D_{11} and $\widehat{D}_{11}$ have strictly positive diagonals and zeroes elsewhere, and where H denotes the Hermitian transpose. Pre-multiplying the constraint (7.2) by $U_{\cdot 1} U_{\cdot 1}^H$ then gives that:

$$\begin{aligned} U_{\cdot 1} U_{\cdot 1}^H \Omega_{22} S \Sigma_\varepsilon \varepsilon_{t+1} &= U_{\cdot 1} U_{\cdot 1}^H Q_{2\cdot}\Pi \eta_{t+1} = U_{\cdot 1} U_{\cdot 1}^H U_{\cdot 1} D_{11} V_{\cdot 1}^H \eta_{t+1} = U_{\cdot 1} D_{11} V_{\cdot 1}^H \eta_{t+1} = Q_{2\cdot}\Pi \eta_{t+1} \\ &= \Omega_{22} S \Sigma_\varepsilon \varepsilon_{t+1} \end{aligned}$$

[28] This equation has a unique solution providing none of the eigenvalues of P are in the set $\left\{ \left| \frac{\omega_{ii}}{\lambda_{ii}} \right| \middle| i \in \{\bar{u}+1, \dots, \dim v_t\} \right\}$, which holds automatically providing the autoregressive process for ε_t is non-explosive. The (non-numerically robust) solution is given by: $\operatorname{vec} S = (\mathrm{P}' \otimes \Omega_{22}^{-1}\Lambda_{22} - I)^{-1} \operatorname{vec} \Omega_{22}^{-1} Q_{2\cdot}\Psi$.

(by the constraint and the unitarity of U). Thus since ε_{t+1} may take the value $\hat{V}_{\cdot 1}\hat{D}_{11}^{-1}\upsilon$ for any υ, by the unitarity of $\hat{V}$, we must have:

$$U_{\cdot 1}U_{\cdot 1}^{H}\hat{U}_{\cdot 1} = \hat{U}_{\cdot 1}. \tag{7.3}$$

This condition is also sufficient for the existence of a solution, which we now demonstrate by exhibiting an explicit solution.

Let $q \coloneqq \operatorname{rank} Q_{2\cdot}\Pi$, so that D_{11} is of dimension $q \times q$. Then following Lubik and Schorfheide (2003), we posit the following set of solutions for the forecast errors η_t:

$$\eta_t = \begin{bmatrix} V_{\cdot 1} & V_{\cdot 2} \end{bmatrix} \begin{bmatrix} D_{11}^{-1}U_{\cdot 1}^{H}\Omega_{22}S\Sigma_\varepsilon \\ M_{\varepsilon,t-1} \end{bmatrix} \varepsilon_t + \begin{bmatrix} V_{\cdot 1} & V_{\cdot 2} \end{bmatrix} \begin{bmatrix} 0 \\ M_{\zeta,t-1} \end{bmatrix} \zeta_t, \tag{7.4}$$

where ζ_t is an arbitrary vector of sunspot shocks, uncorrelated with ε_t, and $M_{\varepsilon,t-1}$ and $M_{\zeta,t-1}$ are arbitrary matrices of size $(\dim \eta_t - q) \times \dim \varepsilon_t$ and $(\dim \eta_t - q) \times \dim \zeta_t$ respectively, known at $t-1$. (The possibility of time variation in $M_{\varepsilon,t-1}$ and $M_{\zeta,t-1}$ was not noticed by Lubik and Schorfheide (2003).) When the condition (7.3) holds, by the unitarity of V we have that:

$$\begin{aligned}
Q_{2\cdot}\Pi\eta_t &= U_{\cdot 1}D_{11}V_{\cdot 1}^{H}\eta_t \\
&= \left(U_{\cdot 1}D_{11}V_{\cdot 1}^{H}V_{\cdot 1}D_{11}^{-1}U_{\cdot 1}^{H}\Omega_{22}S\Sigma_\varepsilon + U_{\cdot 1}D_{11}V_{\cdot 1}^{H}V_{\cdot 2}M_{\varepsilon,t-1}\right)\varepsilon_t + U_{\cdot 1}D_{11}V_{\cdot 1}^{H}V_{\cdot 2}M_{\zeta,t-1}\zeta_t \\
&= U_{\cdot 1}U_{\cdot 1}^{H}\Omega_{22}S\Sigma_\varepsilon\varepsilon_t = U_{\cdot 1}U_{\cdot 1}^{H}\hat{U}_{\cdot 1}\hat{D}_{11}\hat{V}_{\cdot 1}^{H}\varepsilon_t = \hat{U}_{\cdot 1}\hat{D}_{11}\hat{V}_{\cdot 1}^{H}\varepsilon_t = \Omega_{22}S\Sigma_\varepsilon\varepsilon_t
\end{aligned}$$

and so the constraint (7.2) does indeed hold. It is immediate from this solution for the forecast errors that the model has a unique solution if and only if $q = \dim \eta_t$.

In order for there to be a FREE solution, we must be able to express ε_t as a function of η_t and ζ_t. If we pre-multiply the above solution for η_t by $\begin{bmatrix} \Omega_{22}^{-1}U_{\cdot 1}D_{11} & 0 \\ 0 & I \end{bmatrix} V^H$, using condition (7.3) and the unitarity of V we have that:

$$\begin{bmatrix} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1} \end{bmatrix} \varepsilon_t = \begin{bmatrix} \Omega_{22}^{-1}U_{\cdot 1}D_{11} & 0 \\ 0 & I \end{bmatrix} V^H\eta_t - \begin{bmatrix} 0 \\ M_{\zeta,t-1} \end{bmatrix} \zeta_t.$$

Therefore, a FREE solution will certainly exist if $\begin{bmatrix} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1} \end{bmatrix}$ has linearly independent columns for all t, since when this holds, from standard results on the Moore-Penrose pseudo-inverse we have that:

$$\varepsilon_t = \begin{bmatrix} \Sigma_\varepsilon^H S^H S\Sigma_\varepsilon \\ M_{\varepsilon,t-1}^H M_{\varepsilon,t-1} \end{bmatrix}^{-1} \left[\begin{bmatrix} \Sigma_\varepsilon^H S^H \Omega_{22}^{-1}U_{\cdot 1}D_{11} & M_{\varepsilon,t-1}^H \end{bmatrix} V^H\eta_t - M_{\varepsilon,t-1}^H M_{\zeta,t-1}\zeta_t \right]$$

and so it is as if ε_t is in even the limited information set. When $\dim \eta_t - q \geq \dim \varepsilon_t$, $\begin{bmatrix} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1} \end{bmatrix}$ will have linearly independent columns for almost all $M_{\varepsilon,t-1}$.[29] More generally, we require that $\operatorname{rank} S\Sigma_\varepsilon + \dim \eta_t - q \geq \dim \varepsilon_t$.

[29] With $q = 0$, this gives a generalisation of our initial result to the case in which B and Σ_s do not have full rank.

Now by (7.3), $Q_{2\cdot}\Pi V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}^{H} \Omega_{22} S \Sigma_{\varepsilon} = \Omega_{22} S \Sigma_{\varepsilon}$, thus $\operatorname{span} S\Sigma_{\varepsilon} = \operatorname{span} \Omega_{22} S\Sigma_{\varepsilon} \subseteq \operatorname{span} Q_{2\cdot}\Pi$ and so $\operatorname{rank} S\Sigma_{\varepsilon} \leq \operatorname{rank} Q_{2\cdot}\Pi = q$. Thus, if it is to be the case that $\begin{bmatrix} S\Sigma_{\varepsilon} \\ M_{\varepsilon,t-1} \end{bmatrix}$ has linearly independent columns, we must have that:

$$\dim \varepsilon_t - (\dim \eta_t - q) \leq \operatorname{rank} S\Sigma_{\varepsilon} \leq \operatorname{rank} Q_{2\cdot}\Pi = q,$$

which implies $\dim \varepsilon_t \leq \dim \eta_t$. In the special case in which $\dim \varepsilon_t = \dim \eta_t$, these inequalities become equalities, meaning that we must have $\operatorname{span} \Omega_{22} S\Sigma_{\varepsilon} = \operatorname{span} Q_{2\cdot}\Pi$, and hence $\widehat{U}_{\cdot 1} \widehat{U}_{\cdot 1}^{H} U_{\cdot 1} = U_{\cdot 1}$, by (7.3).

The fact that $\begin{bmatrix} S\Sigma_{\varepsilon} \\ M_{\varepsilon,t-1} \end{bmatrix}$ having linearly independent columns implies $\dim \varepsilon_t \leq \dim \eta_t$ makes clear that this condition is not necessary for the existence of a FREE. For example, suppose $\Sigma_{\varepsilon} = 0$, then a FREE must exist independently of the dimension of $\dim \varepsilon_t$ when $M_{\varepsilon,t-1} \equiv 0$.

In order to derive necessary conditions (and tighter sufficient ones) we must first solve for v_t. We begin by pre-multiplying (7.1) by $[I \quad -Q_{1\cdot}\Pi V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}^{H}]$, which gives:

$$\begin{aligned}
&[\Lambda_{11} \quad \Lambda_{12} - Q_{1\cdot}\Pi V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}^{H} \Lambda_{22}] \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} \\
&= [\Omega_{11} \quad \Omega_{12} - Q_{1\cdot}\Pi V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}^{H} \Omega_{22}] \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + (Q_{1\cdot} - Q_{1\cdot}\Pi V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}^{H} Q_{2\cdot})(\mu + \Psi s_t + \Pi \eta_t) \\
&= [\Omega_{11} \quad \Omega_{12} - Q_{1\cdot}\Pi V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}^{H} \Omega_{22}] \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + (Q_{1\cdot} - Q_{1\cdot}\Pi V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}^{H} Q_{2\cdot})(\mu + \Psi s_t) \\
&\qquad + Q_{1\cdot}\Pi V_{\cdot 2} \big(M_{\varepsilon,t-1} \varepsilon_t + M_{\zeta,t-1} \zeta_t \big)
\end{aligned}$$

(using the unitary of U and V, and equation (7.4)).

Hence, if we stack the equation above with the solution for the transversality-violating terms, and pre-multiply by:

$$[Z_{\cdot 1} \Lambda_{11}^{-1} \quad Z_{\cdot 2} - Z_{\cdot 1} \Lambda_{11}^{-1} (\Lambda_{12} - Q_{1\cdot}\Pi V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}^{H} \Lambda_{22})],$$

(valid as Λ_{11} is invertible by construction) we have:

$$\begin{aligned}
v_t = Z_{\cdot 1} \Lambda_{11}^{-1} &[\Omega_{11} Z_{\cdot 1}^{H} + (\Omega_{12} - J\Omega_{22}) Z_{\cdot 2}^{H}] v_{t-1} \\
&+ [Z_{\cdot 1} \Lambda_{11}^{-1} (Q_{1\cdot} - J Q_{2\cdot}) + [Z_{\cdot 2} - Z_{\cdot 1} \Lambda_{11}^{-1} (\Lambda_{12} - J\Lambda_{22})] (\Lambda_{22} - \Omega_{22})^{-1} Q_{2\cdot}] \mu \\
&+ [Z_{\cdot 1} \Lambda_{11}^{-1} (Q_{1\cdot} - J Q_{2\cdot}) \Psi + [Z_{\cdot 2} - Z_{\cdot 1} \Lambda_{11}^{-1} (\Lambda_{12} - J\Lambda_{22})] SP] s_t + Z_{\cdot 1} \Lambda_{11}^{-1} Q_{1\cdot}\Pi V_{\cdot 2} V_{\cdot 2}^{H} \eta_t,
\end{aligned}$$

where Z has been partitioned conformably with w_t and where $J \coloneqq Q_{1\cdot}\Pi V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}^{H}$.

For brevity, we rewrite this solution for v_t as:

$$v_t = T_{-1} v_{t-1} + T_{\mu} + T_s s_t + T_{\eta} \eta_t, \tag{7.5}$$

where T_{-1}, T_{μ}, T_s and T_{η} are defined by matching terms.

Let us assume then that $v_t = \begin{bmatrix} x_t \\ \mathbb{E}_t x_{t+1} \end{bmatrix}$ and $\eta_t = x_t - \mathbb{E}_{t-1} x_t$, as in the general linear expectational model we presented at the start of this appendix. Then if we define $T_{\varepsilon,t-1} \coloneqq T_{\eta} V_{\cdot 2} M_{\varepsilon,t-1}$ and $T_{\zeta,t-1} \coloneqq T_{\eta} V_{\cdot 2} M_{\zeta,t-1}$ and partition all the T. matrices conformably with v_t, we have:

$$\begin{aligned}
\mathbb{E}_t x_{t+1} &= T_{-1,21} x_{t-1} + T_{-1,22} \mathbb{E}_{t-1} x_t + T_{\mu,2} + T_{s,2} s_t + T_{\eta,2} \eta_t \\
&= T_{-1,21} x_{t-1} + T_{-1,22} \mathbb{E}_{t-1} x_t + T_{\mu,2} + T_{s,2} s_t + T_{\varepsilon,t-1,2} \varepsilon_t + T_{\zeta,t-1,2} \zeta_t.
\end{aligned} \tag{7.6}$$

When either $\mathrm{P}=0$, or when s_{t-1} is observed, the feasibility of this solution requires that agents can work out $(T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2})\varepsilon_t$, given knowledge of x_t, η_t and ζ_t. By taking the SVD of $(T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2})$ and $\begin{bmatrix} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1}\end{bmatrix}$ it is straightforward to show that a sufficient condition for feasibility is that:

$$\ker S\Sigma_\varepsilon \cap \ker M_{\varepsilon,t-1} \subseteq \ker\left[T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right], \tag{7.7}$$

in which case:

$$\left[T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right]\varepsilon_t = \left[T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right]\begin{bmatrix} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1}\end{bmatrix}^+ \left[\begin{bmatrix}\Omega_{22}^{-1}U_{\cdot 1}D_{11} & 0 \\ 0 & I\end{bmatrix} V^H \eta_t - \begin{bmatrix} 0 \\ M_{\zeta,t-1}\end{bmatrix}\zeta_t\right],$$

and:

$$\begin{aligned}\mathbb{E}_t x_{t+1} = {} & \left[T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right]\begin{bmatrix} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1}\end{bmatrix}^+ \begin{bmatrix}\Omega_{22}^{-1}U_{\cdot 1}D_{11} & 0 \\ 0 & I\end{bmatrix} V^H x_t + T_{-1,21}x_{t-1} + T_{s,2}\mathrm{P}s_{t-1} \\ & + \left[T_{-1,22} - \left[T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right]\begin{bmatrix} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1}\end{bmatrix}^+ \begin{bmatrix}\Omega_{22}^{-1}U_{\cdot 1}D_{11} & 0 \\ 0 & I\end{bmatrix} V^H\right]\mathbb{E}_{t-1}x_t + T_{\mu,2} \\ & + \left[T_{\zeta,t-1,2} - \left[T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right]\begin{bmatrix} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1}\end{bmatrix}^+ \begin{bmatrix} 0 \\ M_{\zeta,t-1}\end{bmatrix}\right]\zeta_t,\end{aligned}$$

which is in a "semi"-FREE form.

In fact, when $\mathrm{P}=0$, we can provide a more intuitive sufficient condition, under the normalisation that $\Sigma_\varepsilon = I$. In this case, $\ker S\Sigma_\varepsilon = \ker S = \ker Q_{2\cdot}\Psi$ and so for $v \in \ker S\Sigma_\varepsilon \cap \ker M_{\varepsilon,t-1}$, $\Psi v = Q_{1\cdot}^H Q_{1\cdot}\Psi v$ and hence:

$$Q_{1\cdot}^H \Lambda_{11} Z_{\cdot 1}^H \left(T_s\Sigma_\varepsilon + T_{\varepsilon,t-1}\right)v = Q_{1\cdot}^H \Lambda_{11} Z_{\cdot 1}^H Z_{\cdot 1}\Lambda_{11}^{-1}Q_{1\cdot}\left[\Psi + \Pi V\begin{bmatrix} D_{11}^{-1}U_{\cdot 1}^H \Omega_{22} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1}\end{bmatrix}\right]v = \Psi v$$

Hence if $v \in \ker S\Sigma_\varepsilon \cap \ker M_{\varepsilon,t-1} \cap \ker \Psi$, $Q_{1\cdot}^H \Lambda_{11} Z_{\cdot 1}^H \left(T_s\Sigma_\varepsilon + T_{\varepsilon,t-1}\right)v = 0$ which (from pre-multiplying by $[0 \quad I]Z_{\cdot 1}\Lambda_{11}^{-1}Q_{1\cdot}$) implies $\left(T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right)v = 0$. Thus, a sufficient condition for feasibility is that:

$$\ker Q_{2\cdot}\Psi \cap \ker M_{\varepsilon,t-1} \subseteq \ker Q_{2\cdot}\Psi \cap \ker M_{\varepsilon,t-1} \cap \ker \Psi = \ker \Psi \cap \ker M_{\varepsilon,t-1}.$$

Consequently, a sufficient condition for feasibility for any $M_{\varepsilon,t-1}$ is that:

$$\ker Q_{2\cdot}\Psi = \ker \Psi.$$

This states that if there is some linear combination of shocks which does not appear in the transversality-violating block, then that same linear combination does not appear anywhere in the model. This reveals that it is deviations from the saddle path that enable agents to back out the values of shocks.

We now turn to the general case in which we do not assume that $\mathrm{P}=0$ or that s_t is observed even with a lag. Our first claim is that (7.7) is a necessary condition for the existence of a FREE.

Suppose for a contradiction that (7.7) does not hold, but that:

$$\mathbb{E}_t x_{t+1} = \mathcal{R}_{t-1} x_t + \mathcal{S}_{t-1}\zeta_t + \text{other terms known at } t-1,$$

so the expectation can be formed without knowing the value of ε_t. Since $\ker S\Sigma_\varepsilon \cap \ker M_{\varepsilon,t-1} \not\subseteq \ker\left(T_{s,2} + T_{\varepsilon,t-1,2}\right)$, there must exist some $\mathscr{v} \neq 0$ such that $S\Sigma_\varepsilon \mathscr{v} = M_{\varepsilon,t-1}\mathscr{v} = 0$, but $\left(T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right)\mathscr{v} \neq 0$.

Then from (7.6) and the fact that ζ_t is uncorrelated with ε_t, $\mathrm{Cov}_{t-1}(\mathcal{R}_{t-1}x_t, \mathscr{v}\mathscr{v}^H\varepsilon_t | s_{t-1}) = \mathrm{Cov}_{t-1}(\mathbb{E}_t x_{t+1}, \mathscr{v}\mathscr{v}^H\varepsilon_t | s_{t-1}) = \left(T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right)\mathbb{E}_{t-1}\varepsilon_t\varepsilon_t^H \mathscr{v}\mathscr{v}^H = \left(T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,2}\right)\mathscr{v}\mathscr{v}^H \neq 0$. Hence, by our assumption:

$$\begin{aligned} 0 \neq \mathrm{Cov}_{t-1}(\mathcal{R}_{t-1}x_t, \mathscr{v}\mathscr{v}^H\varepsilon_t | s_{t-1}) &= \mathrm{Cov}_{t-1}(\mathcal{R}_{t-1}(\eta_t + \mathbb{E}_{t-1}x_t), \mathscr{v}\mathscr{v}^H\varepsilon_t | s_{t-1}) \\ = \mathbb{E}_{t-1}\,\mathcal{R}_{t-1}\eta_t\varepsilon_t^H\mathscr{v}\mathscr{v}^H &= \mathcal{R}_{t-1}\,\mathbb{E}_{t-1}\left[V\begin{bmatrix} D_{11}^{-1}U_{\cdot 1}^H\Omega_{22}S\Sigma_\varepsilon \\ M_{\varepsilon,t-1}\end{bmatrix}\varepsilon_t + V\begin{bmatrix} 0 \\ M_{\zeta,t-1}\end{bmatrix}\zeta_t\right]\varepsilon_t^H\mathscr{v}\mathscr{v}^H = 0 \end{aligned}$$

(using equation (7.4)), as $S\Sigma_\varepsilon\mathscr{v} = M_{\varepsilon,t-1}\mathscr{v} = 0$ and ζ_t is uncorrelated with ε_t. This gives the required contradiction.

Finally, we show that (7.7) and $\ker T_{s,2} = \{0\}$ are jointly sufficient. First note that if $\ker T_{s,2} = \{0\}$, then $T_{s,2}^+ T_{s,2} = I$. Then, from substituting $\mathbb{E}_{t-1}x_t$ out of the top line of (7.6), using the definition of η_t, subtracting $T_{s,2}\mathrm{P}T_{s,2}^+$ times the equation's lag, then using again the definition of η_t:

$$\begin{aligned} \mathbb{E}_t x_{t+1} = {} & \left[T_{-1,22} + T_{s,2}\mathrm{P}T_{s,2}^+\right]x_t + \left[T_{-1,21} - T_{s,2}\mathrm{P}T_{s,2}^+T_{-1,22}\right]x_{t-1} - T_{s,2}\mathrm{P}T_{s,2}^+T_{-1,21}x_{t-2} \\ & + \left[I - T_{s,2}\mathrm{P}T_{s,2}^+\right]T_{\mu,2} + T_{s,2}\Sigma_\varepsilon\varepsilon_t + \left[T_{\eta,2} - T_{-1,22} - T_{s,2}\mathrm{P}T_{s,2}^+\right]\eta_t \\ & - T_{s,2}\mathrm{P}T_{s,2}^+\left[T_{\eta,2} - T_{-1,22}\right]\eta_{t-1}, \end{aligned}$$

or equivalently (again by the definition of η_t):

$$\begin{aligned} \mathbb{E}_t x_{t+1} = {} & T_{\eta,2}x_t + \left[T_{-1,21} - T_{s,2}\mathrm{P}T_{s,2}^+T_{\eta,2}\right]x_{t-1} - T_{s,2}\mathrm{P}T_{s,2}^+T_{-1,21}x_{t-2} + \left[I - T_{s,2}\mathrm{P}T_{s,2}^+\right]T_{\mu,2} \\ & + \left[T_{-1,22} + T_{s,2}\mathrm{P}T_{s,2}^+ - T_{\eta,2}\right]\mathbb{E}_{t-1}x_t + T_{s,2}\mathrm{P}T_{s,2}^+\left[T_{\eta,2} - T_{-1,22}\right]\mathbb{E}_{t-2}x_{t-1} + T_{s,2}\Sigma_\varepsilon\varepsilon_t. \end{aligned}$$

Hence, since $V_{\cdot 2}^H\eta_t = M_{\varepsilon,t-1}\varepsilon_t + M_{\zeta,t-1}\zeta_t$:

$$\begin{aligned} \mathbb{E}_t x_{t+1} = {} & \left[T_{-1,21} - T_{s,2}\mathrm{P}T_{s,2}^+T_{\eta,2}\right]x_{t-1} - T_{s,2}\mathrm{P}T_{s,2}^+T_{-1,21}x_{t-2} + \left[I - T_{s,2}\mathrm{P}T_{s,2}^+\right]T_{\mu,2} \\ & + \left[T_{-1,22} + T_{s,2}\mathrm{P}T_{s,2}^+ - T_{\eta,2}\right]\mathbb{E}_{t-1}x_t + T_{s,2}\mathrm{P}T_{s,2}^+\left[T_{\eta,2} - T_{-1,22}\right]\mathbb{E}_{t-2}x_{t-1} \\ & + \left[T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right]\varepsilon_t + T_{\zeta,t-1,2}\zeta_t. \end{aligned}$$

By (7.7) then we have the FREE solution:

$$\begin{aligned} \mathbb{E}_t x_{t+1} = {} & \left[T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right]\begin{bmatrix} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1}\end{bmatrix}^+ \begin{bmatrix} \Omega_{22}^{-1}U_{\cdot 1}D_{11} & 0 \\ 0 & I\end{bmatrix} V^H x_t + \left[T_{-1,21} - T_{s,2}\mathrm{P}T_{s,2}^+T_{\eta,2}\right]x_{t-1} \\ & - T_{s,2}\mathrm{P}T_{s,2}^+T_{-1,21}x_{t-2} + \left[I - T_{s,2}\mathrm{P}T_{s,2}^+\right]T_{\mu,2} \\ & + \left[T_{-1,22} + T_{s,2}\mathrm{P}T_{s,2}^+ - T_{\eta,2} - \left[T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right]\begin{bmatrix} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1}\end{bmatrix}^+ \begin{bmatrix} \Omega_{22}^{-1}U_{\cdot 1}D_{11} & 0 \\ 0 & I\end{bmatrix} V^H\right]\mathbb{E}_{t-1}x_t \\ & + T_{s,2}\mathrm{P}T_{s,2}^+\left[T_{\eta,2} - T_{-1,22}\right]\mathbb{E}_{t-2}x_{t-1} + \left[T_{\zeta,t-1,2} - \left[T_{s,2}\Sigma_\varepsilon + T_{\varepsilon,t-1,2}\right]\begin{bmatrix} S\Sigma_\varepsilon \\ M_{\varepsilon,t-1}\end{bmatrix}^+ \begin{bmatrix} 0 \\ M_{\zeta,t-1}\end{bmatrix}\right]\zeta_t, \end{aligned}$$

which establishes the result.

A final remark is that the condition (7.7) holds if and only if:

$$\ker S\Sigma_\varepsilon \cap \ker M_{\varepsilon,t-1} \subseteq \ker T_{s,2}\Sigma_\varepsilon$$

by the definition of $T_{\varepsilon,t-1,2}$. Under determinacy, this in turn holds if and only if $\ker S \subseteq \ker T_{s,2}$.

7.2. E-stability analysis

Following Marcet and Sargent (1989) and Evans and Honkapohja (2001), we calculate the eigenvalues of the Jacobian of the mapping from the PLM (3.1) to the actual law of motion (ALM) (2.3). This mapping takes the form:

$$T\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ b \\ c \\ d_1' \end{bmatrix} = \frac{1}{1-\beta a_1}\begin{bmatrix} \alpha+\rho+\beta(a_2+(b-\rho)a_1) \\ \beta(a_3+(b-\rho)a_2)-\alpha\rho \\ \beta(b-\rho)a_3 \\ \beta(b-\rho)b \\ (1-\rho)\gamma+\beta c(1+b-\rho) \\ \beta(b-\rho)d_1' \end{bmatrix},$$

since:

$$\begin{aligned}(1-\beta a_1)x_{t+1} &= \big(\alpha+\rho+\beta(a_2+(b-\rho)a_1)\big)x_t + (\beta(a_3+(b-\rho)a_2)-\alpha\rho)x_{t-1} \\ &\quad + \beta(b-\rho)a_3x_{t-2} + \beta(b-\rho)b\mathbb{E}_{t-1}x_t + \big((1-\rho)\gamma+\beta c(1+b-\rho)\big) \\ &\quad + \beta(b-\rho)d_1\zeta_t + \sigma\varepsilon_{t+1} + \beta d_1\zeta_{t+1}.\end{aligned}$$

The set of fixed points of T comprises three discrete islands, two of which are single points with $a_3=b=d_1=0$ (i.e. the MSV solutions). These only exist when $\alpha\beta \le \frac{1}{4}$. The third island is of dimension $1+\dim\zeta_t$, capturing the degrees of freedom under indeterminacy.

If we define $\not{f} := \sqrt{\max\{0,1-4\alpha\beta\}}$, then the real-parts of the eigenvalues in the three cases are:

- For the two MSV solutions, indexed by $\lambda \in \left\{\frac{1\pm\not{f}}{2\beta}\right\}$ (and assuming $\alpha\beta \le \frac{1}{4}$):

$$0, \qquad \frac{\beta(1-\rho)}{1-\beta(\rho+\lambda)}, \qquad \frac{\beta\big(\alpha-\rho(1-\beta\rho)\big)}{\big(1-\beta(\rho+\lambda)\big)^2}, \qquad -\frac{\beta\rho}{1-\beta(\rho+\lambda)}$$

- For the sunspot solution (where b is a free parameter):

$$1, \qquad -\frac{b}{\rho-b}, \qquad 1-\frac{1}{\rho-b}, \qquad 1-\frac{|\rho-b|\pm(\rho-b)\not{f}}{2\beta|\rho-b|(\rho-b)}.$$

By the results of Evans and Honkapohja (2001) least squares learning will not converge if any of the eigenvalues' real parts are greater than one. These are similar to, but not identical to, the conditions Evans and Honkapohja (2001), derive for the MSV PLM in their proposition 8.3, under the assumption that the shock is observable.

For convergence in the sunspot case, we at last need the following conditions to hold: $b \le \rho$, $0 \le \rho$, $0 \le \alpha$, $0 < \beta$. Providing these conditions hold, the T map will not have any eigenvalues with real parts greater than one, and those eigenvalues for which the real part equals one will have zero complex parts (a further necessary condition for convergence, without this there may be stable cycles under learning). Note that these parameter restrictions include the most economically relevant case from our motivating example of the Taylor rule, where we would expect $0 \le \rho < 1$, $\alpha = 0$ and $\beta > 0$. However, they also includes many explosive regions (when α is large), and regions exhibiting stable cycles in which ρ is fully identified (i.e. $\alpha\beta > \frac{1}{4}$, which requires large β).

Define $\phi \coloneqq [a_1 \quad a_2 \quad a_3 \quad b \quad c \quad d_1]'$. The system is weakly e-stable at the solution $\left[\frac{1}{\beta} + (\rho - \tilde{b}) \quad -\frac{\alpha+\rho}{\beta} \quad \frac{\alpha\rho}{\beta} \quad \tilde{b} \quad -\frac{\gamma(1-\rho)}{\beta} \quad \tilde{d}\right]'$ for fixed $\tilde{b}$ and $\tilde{d}$ if and only if the differential equation $\dot{\phi} = T\phi - \phi$ is locally stable at this solution, where the dot denotes a derivative with respect to "virtual-time" τ.

Defining:

$$\psi \coloneqq \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left[\phi - \begin{bmatrix} \frac{1}{\beta} + (\rho - \tilde{b}) \\ -\frac{\alpha+\rho}{\beta} \\ \frac{\alpha\rho}{\beta} \\ \tilde{b} \\ -\frac{\gamma(1-\rho)}{\beta} \\ \tilde{d}' \end{bmatrix} \right] = \begin{bmatrix} a_1 - \frac{1}{\beta} - (\rho - b) \\ a_2 + \frac{\alpha+\rho}{\beta} \\ a_3 - \frac{\alpha\rho}{\beta} \\ b - \tilde{b} \\ c + \frac{\gamma(1-\rho)}{\beta} \\ d_1' - \tilde{d}' \end{bmatrix},$$

we then have that:

$$\dot{\psi} = -\frac{1}{\psi_1 - \psi_4 + (\rho - \tilde{b})} \begin{bmatrix} \psi_2 + \psi_1\left(\psi_1 + \frac{1}{\beta} + \rho\right) \\ \psi_3 + \psi_1\left(\psi_2 - \frac{\alpha+\rho}{\beta}\right) \\ \psi_1\left(\psi_3 + \frac{\alpha\rho}{\beta}\right) \\ \psi_1(\psi_4 + \tilde{b}) \\ \psi_5 + \psi_1\left(\psi_5 - \frac{\gamma(1-\rho)}{\beta}\right) \\ \psi_1(\psi_6 + \tilde{d}') \end{bmatrix}.$$

Combining the third and fourth equations then gives that:

$$\frac{\psi_4(\tau) + \tilde{b}}{\psi_3(\tau) + \frac{\alpha\rho}{\beta}} = \frac{\psi_4(0) + \tilde{b}}{\psi_3(0) + \frac{\alpha\rho}{\beta}} = \frac{b(0)}{a_3(0)}.$$

Using this equation, we can substitute ψ_4 out of the above differential equation. We can also ignore the final equation since it is the only one containing ψ_6, meaning that if the other components converge to something, so will ψ_6. The resulting four-equation system has real eigenvalues components:

$$\frac{\beta}{C\alpha - \beta}, \qquad \frac{\beta}{\rho(C\alpha - \beta)}, \qquad \frac{1 \pm \sqrt{\max\{0, 1 - 4\alpha\beta\}}}{2\rho(C\alpha - \beta)}$$

when evaluated at the (zero) steady-state, where $C \coloneqq \frac{b(0)}{a_3(0)}$. Given the necessary conditions already derived ($\tilde{b} \leq \rho$, $0 \leq \rho$, $0 \leq \alpha$ and $0 < \beta$), for these real eigenvalues components to be strictly negative, we require that $\alpha C - \beta \leq 0$. However, since we only require local convergence, we may assume that b and a_3 begin close enough to their steady state for us to have $C = \frac{\beta \tilde{b}}{\alpha \rho} + \beta \epsilon$ for some ϵ, small in magnitude. Then $\alpha C - \beta \leq 0$ if and only if $\tilde{b} \leq \rho(1 - \alpha \epsilon)$. We can always find an ϵ for which this holds (i.e. start sufficiently close to the solution) providing $\tilde{b} < \rho$ or $\alpha = 0$ and $\tilde{b} \leq \rho$.

We now turn to the second PLM, (3.2). Since the two PLMs only differ in a term that is unknown at t, period t expectations of x_{t+1} are identical under both PLMs, meaning that the T-map is just as before, but with one extra component, taking d_0' to $\frac{\beta}{1-\beta a_1} d_1'$. Consequently, a solution is weakly (strongly) e-stable under the PLM (3.2) if and only if it is weakly (strongly) e-stable under the PLM (3.1).[30]

[30] This follows from integrating the corresponding differential equation, to give $d_0(\tau) = e^{-\tau} \int_0^{\tau} \frac{\beta}{1-\beta a_1(t)} d_1(t) e^t \, dt + d_0(0) e^{-\tau}$. Hence as $\tau \to \infty$, $d_0(\tau) \to \lim_{t \to \infty} \frac{\beta}{1-\beta a_1(t)} d_1(t)$.