Business cycles in space*

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Abstract

Many important shocks in the real world are correlated not only in time, but also across some notion of space. This may be physical space, or the space of product, firm or household types. As a result of this spatial correlation, aggregate volatility emerges naturally from idiosyncratic shocks. In this paper, we introduce a tractable framework that allows for such shocks without necessitating the discretisation of space, or a departure from perturbation approximation. As a lead example, we construct a dynamic, stochastic, general equilibrium model of economic geography (DSGEEG). This model features population movement, firm dynamics and semi-endogenous growth. Using it, we show how transitory, spatially located productivity shocks can lead to persistent movements in population, helping to explain internal migration patterns in the U.S., and regional wage and investment dynamics. As an additional theoretical contribution, we derive conditions for the existence of continuous-in-space shock processes on a range of spaces of economic interest.

JEL: E3, F4, C31

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1 Introduction

Modern dynamic macroeconomic models are driven by a variety of shocks, including shocks to productivity, discounting and the disutility of labour supply. Each of these is likely to be spatially correlated. Good ideas diffuse in the local neighbourhood, leading to spatial correlation in productivity.[1] A desire to bring forward consumption, or to reduce labour supply might be driven by the weather, which is highly spatially correlated.[2] In this paper we study the implications of productivity shocks that are correlated across geography on macroeconomic dynamics. To do so, we introduce a tractable framework that allows for shocks that are continuous in space, and hence correlated over it, using it to build a DSGE model incorporating the key features of the new economic geography literature. Regional heterogeneity is driven by a continuous-in-space productivity process which is disciplined by estimating the spatial correlation of productivity shocks in the U.S. This allows us to see how spatially located productivity shocks can drive movements in population, helping to explain migration patterns in the U.S., and linking business cycle shocks to lower frequency changes in economic activity across geography.

Modelling the macroeconomy without considering the geographical dispersion of economic conditions is likely to hide the true welfare costs of business cycles. For example, Figure 1 plots the dispersion of county level wage growth over recent decades. The observed volatility of county-level wage growth is unheard of at the aggregate level and points to much larger welfare costs than those implied by aggregate business cycles. In addition to studying these normative issues, our framework opens up the possibility for exploring the effects this heterogeneity can have on the propagation of macroeconomic shocks and policy changes, and for examining the macroeconomic consequences of local shocks.

Our model features the key ingredients highlighted by the new economic geography literature (see e.g. Krugman (1998) or Redding (2013) for reviews). Firstly, firms in the model produce differentiated varieties, and consumers have a taste for variety. Thus, in regions with high population, since there will be greater firm entry, there will also be greater productivity, via the variety effect. We model firm entry here following Bilbiie et al. (2012). Since high population is associated with high productivity, high population regions are attractive to further inward migration. Concentration of population also reduces transport costs, further increasing effective productivity, particularly since we allow for a role for intermediates in production, following Krugman and Venables (1995).

These agglomeration forces are counterbalanced by the populations' need for living space, and their

[1] For example, firms located physically closer to frontier firms have been shown to catch up quicker (Griffith et al., 2009; Comin et al., 2012; Cardamone, 2017), and physical distance is found to be more important than economic distance in the spill-over of productivity (Glass et al., 2013).

[2] Shocks may also be correlated across other notions of space, such as the space of product, firm or household types, or, more generally, any space of economic agents in which "nearby" agents are expected to share similar properties and experience similar shocks. For example, one might expect firms producing similar products to experience correlated shocks to their returns from R&D. As an additional contribution, in appendix A, we derive conditions for the existence of such processes for a range of spaces of economics interest.

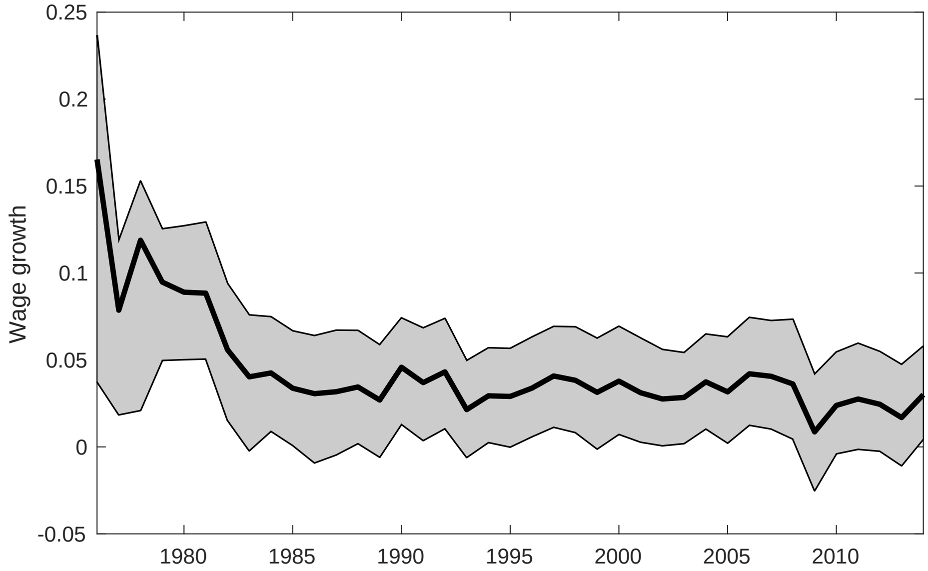

Figure 1: U.S. county year-on-year growth in average weekly wage: mean & 10th/90th percentiles (unweighted). Source: Bureau of Labor Statistics

need to consume agricultural goods. As population increases in a location, more of it must be allocated for living space, so there is less remaining to produce the agricultural good. Consequently, more of the agricultural good must be imported into the location, pushing up its relative price. This increases the desirability of locations producing significant quantities of the agricultural good. Additionally, an increase in productivity pushes up wages, making entry relatively more expensive in high population locations. While these dispersion forces are enough for the existence of a steady-state, it will turn out that they are insufficient for the local stability of that steady-state. Small, temporary, exogenous changes in productivity in one location can drive a move to other long-run equilibria. To counter this, we introduce an additional dispersive force: a preference for living in a location with moderate population. We calibrate the strength of this force so that the model is only just locally stable. Under this calibration, positive productivity shocks have extremely persistent effects on population. In essence, the location that gets "lucky" originally will have a permanent advantage.

The existing literature contains many techniques for solving heterogeneous agent models in which shocks are i.i.d. across agents. These generally necessitate time-consuming global solution methods (see e.g. Den Haan, 2010). Of course, one way of taking a local approximation to a heterogeneous agent model is to solve the model with a finite collection of agents. However, getting reasonable accuracy along these lines requires a prohibitively high number of agents, given the $\frac{1}{\sqrt{N}}$ rate guaranteed by the central limit theorem. In our modelling framework however, due to the continuity and bounded variation of the driving stochastic processes, accurate solution does not require large state spaces. This makes taking a perturbation solution to the model much more tractable, which is the approach we pursue here.[3]

[3] As pointed out by Desmet and Rossi-Hansberg (2014) "Incorporating a continuum of locations into a dynamic framework is a challenging task for two reasons: it increases the dimensionality of the problem by requiring agents to understand the distribution of economic activity over time and over space, and clearing goods and factor markets is complex because

1.1 The puzzle of aggregate volatility

Since spatial correlation allows idiosyncratic shocks to lead to aggregate volatility, it gives a partial answer to the question of the sources of aggregate fluctuations. The standard puzzle is as follows: suppose an economy comprises N firms, each of which receive an i.i.d. productivity draw with finite variance. Then by the Lindeberg–Lévy CLT, as $N \to \infty$, the standard deviation of aggregate productivity declines as $\frac{1}{\sqrt{N}}$. Modern economies have millions of firms, so a back of the envelope calculation suggests they ought to have a miniscule variance. Our solution to the puzzle is almost trivial, we just assume that firms receive correlated shocks, as indeed they do in the real world. While perhaps simplistic, this story both captures the micro-level data on correlation, resolves some macro-level puzzles, and provides a technical approach for addressing important macroeconomic issues.

This paper's story of the source of aggregate variation is a complement to those of Gabaix (2011) and Acemoglu et al. (2012). Gabaix (2011) argues that aggregate fluctuations may be explained if the distribution of firm sizes is fat-tailed so the economy contains large firms and firms receive a multiplicative productivity shock with magnitude unrelated to their size., Gabaix shows that in the extreme case of a s^{-1} tail to the firm size distribution, aggregate volatility declines as $\frac{1}{\log N}$, meaning that the observed aggregate volatility need not be particularly surprising. However, this explanation can at best be part of the story, as the law of large numbers applies just as well within a large firm (comprised of many workers, in many factories, producing many different products or components) as it does across firms. So, we ought to be as surprised that the variance of productivity does not wash out in large firms, as we are that it does not wash out in aggregate. Gabaix suggests that the units that make up a firm may themselves follow a power-law size distribution, but the justification for this is unclear: if firms or firm units are receiving shocks with a second moment, it is hard to see how a power-law size distribution could emerge in the first place, given such a distribution has infinite second moment. Our model gives one way of completing Gabaix's story: if the many products produced by large firms are all nonetheless close in product space, then they will be tightly correlated, and there will still be substantial variance at the firm level.

In our model, spatially correlated productivity shocks drives the geographic variation in economic conditions. In addition to the empirical evidence we present here, there has been a large literature finding support for such spatial correlation. For example, Griffith et al. (2009) show that firms that are physically closer to frontier firms catch-up quicker than those further away (see also Comin et al., 2012). Glass et al. (2013) find that physical distance is more important than economic distance in the cross-border spill-over of technology, and Cardamone (2017), studying Italian R&D, again finds, amongst several fac-

prices depend on trade and mobility patterns. These two difficulties typically make spatial dynamic models intractable, both analytically and numerically." That our approach enables us to solve rich dynamic spatial models without drastic simplification is a considerable advantage to our approach.

tors, physical closeness is important for the spread of productivity improvements (see also Glass et al., 2016).

1.2 Other related literature

There have been several prior dynamic models of economic geography, though these have usually been non-stochastic, with discrete space and simplifications which remove any forward-looking component to economic decisions. Examples preserving some dynamic component to decisions include the model of Caselli and Coleman II (2001) and Eckert and Peters (2018), who build OLG models to explain structural change featuring capital investment decisions. Models with a static or purely backward looking solution include Michaels et al. (2012), and Nagy (2016). Examples with continuous space include Desmet and Rossi-Hansberg (2014) and Desmet et al. (2018), who propose models which, although dynamic, have a solution that is backwards looking. Examples of dynamic economic geography models with a stochastic component include Duranton (2007), who presents a version of the Grossman and Helpman (1991) model with a fixed discrete set of cities, and Rossi-Hansberg and Wright (2007), who analyse a model with many point cities to match the city size distribution. In both cases, restrictive assumptions are made to ensure tractability.

There have been a few simple non-stochastic macroeconomic models involving continuous space, but without allowing for substantial interactions between locations. These include Brito (2004) and Boucekkine et al. (2009), who present Ramsey models with continuous space, as well as Quah (2002), who presents a version of the Lucas (1988) model on the surface of a sphere.

Relative to the aforementioned papers, we are almost unconstrained in our model building. We will have continuous space, with a distribution of population over the space that is changing over time in response to varying regional opportunities. We will have both many state variables, and many forward-looking variables, and we will need to impose transversality constraints in solving the model. We also allow for a rich shock structure, with both spatially located shocks, and aggregate shocks.

1.3 Outline

The structure of our paper is follows. In Section 2, we introduce spatially correlated shock processes, and discuss the tools we have produced to assist with the simulation of such models. The theoretical results on existence of continuous stochastic processes are relegated to Appendix A, due to their technical nature. In Section 3, we describe our dynamic stochastic general equilibrium model of economic geography (DSGEEG). Section 4 presents theoretical and computational results from this model, and Section 5 concludes.

2 Spatially correlated shock processes

In macroeconomics, it is typical to work with continuums of types, as it ensures individuals have no impact on aggregates. Given this, in order for nearby types to receive correlated shock draws, it is sufficient that the drawn shock is continuous in type space.

As an example, to fix ideas, suppose that firms produce products of types indexed by $[0,1]$. We would expect firms producing similar products to receive similar productivity shocks. We might then suppose that if $a_{x,t}$ is the log-productivity of firm $x \in [0,1]$ at t, then:

$$a_{x,t} = (1-\rho)\,\mu + \rho a_{x,t-1} + \sigma\varepsilon_{x,t},$$

where $\varepsilon_{x,t}$ is a continuous function of x. By induction, it is then trivial to show that $a_{x,t}$ is a continuous function of x as well, so firm productivity is always spatially correlated. Later we will consider generalisations of this structure in which productivity today can depend on the lagged productivity of nearby firms as well.

2.1 Simple examples

If we wanted a discrete time stationary stochastic process, using a Gaussian AR (1) process would be the natural choice. Ornstein-Uhlenbeck processes are the continuous equivalent of Gaussian AR (1) processes, and are defined on $\mathbb{R}$. Using a draw from an Ornstein-Uhlenbeck process is one possibility when we want shocks on e.g. the type space $[0,1]$. These processes are characterised by Gaussian marginals, with an auto-covariance function of the form:

$$\operatorname{cov}(\varepsilon_x, \varepsilon_{\tilde{x}}) \; = \sigma^2 \exp\{-\zeta\,|x-\tilde{x}|\}\ ,$$

where σ^2 scales the variance, and $\zeta > 0$ controls the persistence. An example of a realisation of such a process is given in Figure 2. As $\zeta \to 0$, we get Brownian motion, and as $\zeta \to \infty$ we get "white-noise". It turns out that Ornstein-Uhlenbeck processes are the unique stationary, Gaussian, Markovian process on $\mathbb{R}$ (Doob, 1942). The other processes we look at in Appendix A will not be Markovian, but whereas the Markovian assumption is natural in time, in space it does not have any particular intuitive appeal.

One downside to using $[0,1]$ as the type space is that types at the end of the interval may end up with different properties. For example, if space is physical space, and some goods are produced at each location, then households at the end of the interval will have to pay higher transport costs. It is often convenient then to work with spaces which are invariant under translation, since this will ensure that all points are a priori the same. One way to do this is to work with circles, spheres or torii. Recall that a

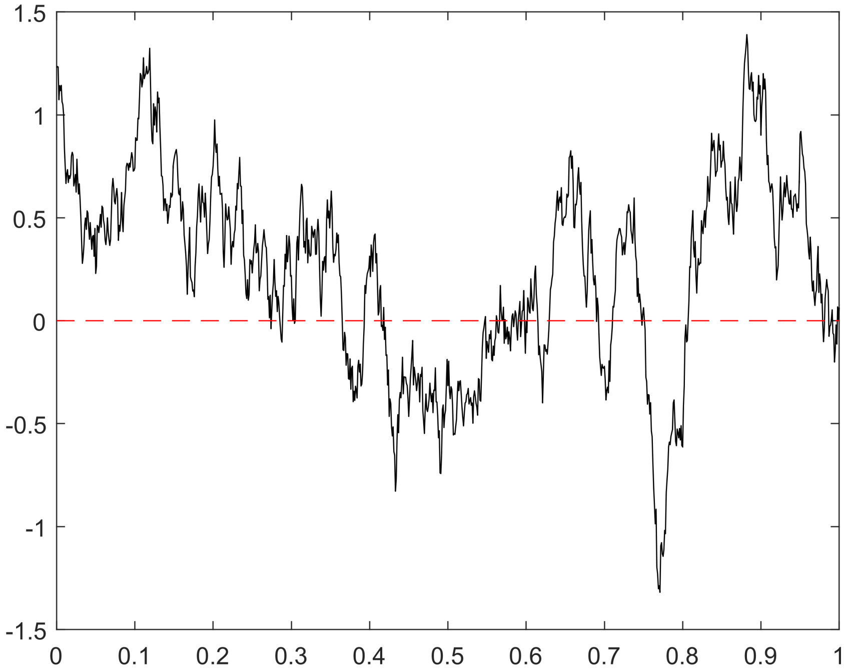

Figure 2: A realisation of an Ornsein-Uhlenbeck process with $\boldsymbol{\sigma = 1}$ and $\boldsymbol{\zeta = 8}$

torus is a "donut" shape. It may be thought of as a square in which when you move off the top edge, you reappear on the bottom edge, and when you move off the left edge, you reappear on the right.

It turns out that the natural Gaussian continuous stochastic process on a circle or a torus is characterised by an auto-covariance function of the form:

$$\operatorname{cov}\left(\varepsilon_x, \varepsilon_{\tilde{x}}\right) \ = \sigma^2 s\left(\zeta, d\left(x, \tilde{x}\right)\right),$$

where σ^2 scales the variance, $\zeta > 0$ controls the persistence, d is the distance function (metric) being used on the circle (identified with $[0,1]$) or torus (identified with $[0,1] \times [0,1]$) and for all $\zeta, d > 0$:

$$s\left(\zeta, d\right) = \frac{\exp\left(-\zeta d + \zeta \overline{d}\right) \ + \exp\left(\zeta d - \zeta \overline{d}\right)}{\exp\left(\zeta \overline{d}\right) \ + \exp\left(-\zeta \overline{d}\right)},$$

where:

$$\overline{d} \equiv \sup_{x, \tilde{x} \in X} d\left(x, \tilde{x}\right)$$

is the maximum distance between points. Further examples, along with proofs that these processes are well-defined are given in Appendix A.

2.2 Simulating DSGE models with continuous in space stochastic processes

Models with continuous-in-space stochastic processes, like all the models we are interested in, possess an infinite dimensional state, making simulation non-trivial. However, by the continuity of the shock, all variables including the states are continuous in space. Furthermore, it is easy to show that they are all of bounded variation. Consequently, their integrals may be approximated arbitrarily well by their values at finitely many points, via standard quadrature methods, and convergence of this quadrature will be much faster than with Monte Carlo (as used say in the Krusell-Smith algorithm). For example, if the trapezium rule is used on a circle, then the error will fall as $\mathrm{O}\left(\frac{1}{n^2}\right)$ (Rahman and Schmeisser, 1990), rather than $\mathrm{O}\left(\frac{1}{\sqrt{n}}\right)$ with Monte Carlo.

These results carry across to convergence in distribution. For example, if we let $x \mapsto \varepsilon_{x,n}$ be the stochastic process on the circle in which $\left[\varepsilon_{\frac{a}{n},n}\right]_{a=1,\dots,n}$ is jointly normally distributed with mean zero and covariance $\operatorname{cov}\left(\varepsilon_{\frac{a}{n},n}, \varepsilon_{\frac{b}{n},n}\right) = \exp\left[-\zeta d\left(\frac{a}{n}, \frac{b}{n}\right)\right]$, and where $\varepsilon_{x,n}$ is given by linear interpolation away from these points, and we let $x \mapsto \varepsilon_x$ be the Gaussian stochastic process on the circle with mean zero and covariance $\operatorname{cov}\left(\varepsilon_{x,t}, \varepsilon_{\tilde{x},t}\right) = \exp\left[-\zeta d\left(x, \tilde{x}\right)\right]$, then it is a theorem (Pedersen, 2002) that $\left[\varepsilon_{x,n}\right]_{x\in[0,1]}$ converges in distribution to $\left[\varepsilon_{x}\right]_{x\in[0,1]}$ as $n \to \infty$, uniformly in x.

Our approach to simulation then is to choose a regular grid of points in space, and then to approximate the value of endogenous variables at points off this grid by linear interpolation. Having fixed the grid, we can then solve the model by standard methods for finite dimensional models; indeed, we may even use Dynare (Adjemian et al., 2011).[4]

3 A dynamic stochastic general equilibrium economic geography model

We now present our DSGE model of economic geography. This model combines standard real business cycle features, with features from the workhorse models of new economic geography of e.g. Krugman (1991). The work of Bilbiie et al. (2012) on firm dynamics is used to help bridge the gap between these two literatures. We are careful in our modelling choices to ensure that the model is consistent with balanced growth. This rules out non-homothetic preferences, for example.

The model features two types of final goods, agricultural products and manufactured products. Manu-

[4]Of course, manually adding equations for every point on the grid would be extremely time consuming. Luckily though, Dynare provides a pre-processor language that enables one to loop over points. To assist further with the creation of spatial models, we provide a Dynare toolkit that can automatically define spatially correlated shock processes, including ones with spatial diffusion. It is available under an open source license from: `https://github.com/tholden/DynareTransformationEngine`. The model presented in this section is contained in "ExampleWithSpatialShocks.mod" in that repository. For a more complete example of all the capabilities of this toolkit, see the code for this paper's main model here: `https://github.com/tholden/DynamicSpatialModel`.

factured products are an aggregate of differentiated varieties produced by the firms in the model. Both these differentiated varieties and agriculture are produced using raw goods as an input, where raw goods are produced from capital, labour and intermediate inputs of manufactured goods. These raw goods may be thought of as providing production services. They are introduced chiefly to avoid complicating the model with multiple varieties of capital and labour. In agricultural production, the raw good is combined with land, whereas it is the sole input in the production of manufactured goods. For tractability, agricultural goods will be freely transportable and tradeable across locations, and manufactured goods will be untradeable. The differentiated varieties will be tradeable however, and will be subject to iceberg transportation costs.

Firms, capital and population will all have a density over space. We denote the set of points in space by X and assume that land is uniformly distributed over X. We normalise the total measure of X to 1, so $\int_X \mathrm{d}x = 1$. We assume that a metric is defined on X, giving the distance between any $x, \tilde{x} \in X$ as $d(x, \tilde{x})$.

At each location, there will be a representative household. For simplicity though, we assume that all these households are part of one representative family, and that household decisions are coordinated by a family head, who maximises a utilitarian social welfare function. As usual, this is equivalent to assuming the existence of complete markets between households. While assuming complete markets may be a little of a stretch, it greatly enhances the tractability of our model. Without this assumption, at each point in space there would be a distribution of asset holdings, as households who moved to that location would come with different assets to those who were already there. Furthermore, the decision of a household on where to move would be complicated by the need to consider what their utility would be at some location, which will differ in general from the utility of the households already there. If the reader is sceptical of the existence of complete markets in reality, it may help to think of our assumptions as giving the outcomes that a social planner could achieve with sufficient instruments. If real government policy is sufficiently close to optimal, then our model will provide a good guide to real world outcomes.

3.1 Manufactured good aggregator at $x \in X$

The non-tradeable manufactured final good at location x is produced by a perfectly competitive industry with access to the CES production function:

$$Y_{x,t} = \left[\int_X \int_0^{J_{\tilde{x},t}} \left(\frac{Y_{j,\tilde{x},x,t}}{\exp\left[\tau_t d\left(x, \tilde{x}\right)\right]} \right)^{\frac{1}{1+\lambda}} \mathrm{d}j \mathrm{d}\tilde{x} \right]^{1+\lambda}. \tag{3.1}$$

Here, $\frac{1+\lambda}{\lambda}$ is the elasticity of substitution between varieties, τ_t gives the strength of iceberg transportation costs in period t, $J_{\tilde{x},t}$ gives the mass of firms located at $\tilde{x}$ in period t, and $Y_{j,\tilde{x},x,t}$ denotes the quantity of the differentiated variety produced by firm j at $\tilde{x}$ that is used in producing the final manufactured good at location x in period t. For convenience, we relabel firms each period, so that even with firm exit, the measure of firms located at $\tilde{x}$ in period t is still given by $[0, J_{\tilde{x},t}]$. Allowing for imperfect substitutability between varieties is crucial both because it allows for an increase in the measure of firms to increase productivity, providing an agglomerative force, and because it allows for the introduction of transport costs without having to deal with positivity constraints on consumption of varieties. Transport costs provide further agglomerative pressure, since in locations producing large numbers of varieties, less will need to be spent on transport.

In period t, the manufactured final good at x is sold at a price $P_{x,t}$, and the input produced by firm j in location $\tilde{x}$ is sold at a price $P_{j,\tilde{x},t}$. Thus, the profits of firms making the final manufactured good are given by:

$$P_{x,t}Y_{x,t} - \int_X \int_0^{J_{\tilde{x},t}} P_{j,\tilde{x},t}Y_{j,\tilde{x},x,t}\mathrm{d}j\mathrm{d}\tilde{x}.$$

From the first order condition for $Y_{j,\tilde{x},x,t}$ we then have that:

$$Y_{j,\tilde{x},x,t} = Y_{x,t}\left(\frac{P_{x,t}}{P_{j,\tilde{x},t}}\right)^{\frac{1+\lambda}{\lambda}} \exp\left[-\frac{\tau_t}{\lambda}d\left(x,\tilde{x}\right)\right], \tag{3.2}$$

so demand is decreasing in the distance to the seller of the variety in question. From substituting equation (3.2) into equation (3.1), we have that:

$$P_{x,t} = \left[\int_X \int_0^{J_{\tilde{x},t}} \left(P_{j,\tilde{x},t}\exp\left[\tau_t d\left(x,\tilde{x}\right)\right]\right)^{-\frac{1}{\lambda}}\mathrm{d}j\mathrm{d}\tilde{x}\right]^{-\lambda}. \tag{3.3}$$

Furthermore, equation (3.2) implies that the total demand for the good produced by firm j in location x at t is given by:

$$Y_{j,x,t} \equiv \int_X Y_{j,x,\tilde{x},t}\mathrm{d}\tilde{x} = P_{j,x,t}^{-\frac{1+\lambda}{\lambda}}\overline{Y}_{x,t}, \tag{3.4}$$

where:

$$\overline{Y}_{x,t} \equiv \int_X Y_{\tilde{x},t}P_{\tilde{x},t}^{\frac{1+\lambda}{\lambda}} \exp\left[-\frac{\tau_t}{\lambda}d\left(\tilde{x},x\right)\right]\mathrm{d}\tilde{x}. \tag{3.5}$$

3.2 Firms at $x \in X$

The j^{th} firm at location x producing a differentiated variety has access to the production function:

$$Y_{j,x,t} = Z_{j,x,t}, \tag{3.6}$$

where $Z_{j,x,t}$ is the amount of the raw good ("production services") it purchases in period t, at a price of $\mathcal{P}_{x,t}$. The firm maximises its profits which are given by:

$$Y_{j,x,t}\left(P_{j,x,t} - \mathcal{P}_{x,t}\right) = \left(P_{j,x,t}^{-\frac{1}{\lambda}} - \mathcal{P}_{x,t} P_{j,x,t}^{-\frac{1+\lambda}{\lambda}}\right) \overline{Y}_{x,t}.$$

From the first order condition for $P_{j,x,t}$, we derive the usual mark-up pricing condition:

$$P_{j,x,t} = (1+\lambda)\,\mathcal{P}_{x,t}. \tag{3.7}$$

Consequently, profits are equal across firms located at x in period t, and are given by:

$$\Pi_{x,t} \equiv \frac{\lambda}{1+\lambda}(1+\lambda)^{-\frac{1}{\lambda}} \mathcal{P}_{x,t}^{-\frac{1}{\lambda}} \overline{Y}_{x,t}.$$

Furthermore, from substituting equation (3.7) into equation (3.3) we have that:

$$P_{x,t} = (1+\lambda)\left[\int_X J_{\tilde{x},t}(\mathcal{P}_{\tilde{x},t} \exp\left[\tau_t d\left(x, \tilde{x}\right)\right]\,)^{-\frac{1}{\lambda}} \mathrm{d}\tilde{x}\right]^{-\lambda}.$$

Much as in the model of Bilbiie et al. (2012), firm entry requires paying ϕ_t units of the raw input, and firms exit at an exogenous rate, δ_J. Since firms are owned by the representative family, they discount the future with that family's stochastic discount factor, which we denote by Ξ_{t+1}. This leads to the free entry condition:

$$\phi_t \mathcal{P}_{x,t} = \mathbb{E}_t \sum_{s=0}^{\infty} \left[\prod_{k=1}^{s} \Xi_{t+k}\right] (1-\delta_J)^s \Pi_{x,t+s},$$

i.e.:

$$\phi_t \mathcal{P}_{x,t} = \Pi_{x,t} + (1-\delta_J)\,\mathbb{E}_t \Xi_{t+1} \phi_{t+1} \mathcal{P}_{x,t+1}. \tag{3.8}$$

3.3 Capital holding company at $x \in X$

Without loss of generality, we assume that the capital stock at location x is owned by a representative capital holding company that is located there. The capital stock at x evolves according to:

$$K_{x,t} = (1 - \delta_K) K_{x,t-1} + \left[1 - \Phi\left(\frac{I_{x,t}}{I_{x,t-1}}\right)\right] I_{x,t}, \tag{3.9}$$

where δ_K is the depreciation rate of capital, and Φ reflects Christiano et al. (2005) style investment adjustment costs, with $\Phi(1) = \Phi'(1) = 0$ and $\Phi''(1) > 0$. Capital is rented out at a rate $\mathcal{R}_{K,x,t}$ per unit at location x in period t and is immovable across locations. Including investment adjustment costs ensures that it is hard to move capital across locations by disinvesting in one location and reinvesting somewhere else. It thus helps to give persistence to the location of clusters of economic activity ("cities").

The representative capital holding company at x chooses period t investment to maximise their profits:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left[\prod_{k=1}^{s} \Xi_{t+k}\right] (\mathcal{R}_{K,x,t+s} K_{x,t+s-1} - P_{x,t} I_{x,t+s})$$

subject to law of motion for capital, equation (3.9). Writing $Q_{x,t}$ for the Lagrange multiplier on equation (3.9), this leads to the first order condition for $K_{x,t}$:[5]

$$1 = \mathbb{E}_t \Xi_{t+1} \frac{\mathcal{R}_{K,x,t+1} + Q_{x,t+1}(1 - \delta_K)}{Q_{x,t}},$$

and first order condition for $I_{x,t}$:

$$P_{x,t} = Q_{x,t}\left(1 - \Phi\left(\frac{I_{x,t}}{I_{x,t-1}}\right) - \Phi'\left(\frac{I_{x,t}}{I_{x,t-1}}\right)\frac{I_{x,t}}{I_{x,t-1}}\right) + \mathbb{E}_t \Xi_{t+1} Q_{x,t+1} \Phi'\left(\frac{I_{x,t+1}}{I_{x,t}}\right)\left(\frac{I_{x,t+1}}{I_{x,t}}\right)^2.$$

3.4 Agriculture at $x \in X$

The agricultural sector at location x is perfectly competitive and has access to the production function:

$$F_{x,t} = L_{x,t}^{\gamma} Z_{F,x,t}^{1-\gamma},$$

where $Z_{F,x,t}$ is the amount of the raw good ("production services") used as an input to farming at location x in period t, and where $L_{x,t}$ is the amount of land allocated to farming at location x in period t. Farm land $L_{x,t}$ is rented from households at a rate of $\mathcal{R}_{L,x,t}$ per unit, and, as before, the raw good costs $\mathcal{P}_{x,t}$ per unit. We take the agricultural product as our numeraire (i.e. we assume it has unit cost), and further assume that it is tradeable without costs. The assumption of costless trade in agricultural

[5] The Lagrangian for this problem is contained in Appendix C.1.

products is common in the new economic geography literature previously cited. Introducing trade costs to the agricultural product would have required either introducing differentiation in agricultural products, or dealing with positivity constraints on agricultural production at each location, both of which would have substantially complicated the model. Furthermore, it is plausible that agricultural products should have relatively low trade costs compared to the rest of economic output, since many non-agricultural products are essentially non-tradeable (consider e.g. services, which, in our model, is subsumed within manufacturing).

Firms producing the agricultural good at location x in period t have profits:

$$F_{x,t} - \mathcal{R}_{L,x,t} L_{x,t} - \mathcal{P}_{x,t} Z_{F,x,t},$$

and thus choose $L_{x,t}$ such that:

$$\gamma \frac{F_{x,t}}{L_{x,t}} = \mathcal{R}_{L,x,t}, \tag{3.10}$$

and $Z_{F,x,t}$ such that:

$$(1-\gamma) \frac{F_{x,t}}{Z_{F,x,t}} = \mathcal{P}_{x,t}. \tag{3.11}$$

3.5 Raw good production at $x \in X$

The raw good at location x is produced in period t by a perfectly competitive industry with access to the production function:

$$Z_{x,t} = \left[K_{x,t-1}^{\alpha} (A_{x,t} H_{x,t})^{1-\alpha} \right]^{1-\kappa} M_{x,t}^{\kappa},$$

where in period t capital $K_{x,t-1}$ is rented from capital holding companies at a rate of $\mathcal{R}_{K,x,t}$ per unit, labour $H_{x,t}$ is hired from the household at a wage $W_{x,t}$ per unit, and intermediate inputs of the final manufactured good, $M_{x,t}$, cost $P_{x,t}$ per unit. $A_{x,t}$ is productivity at location x in period t. Allowing for capital in production is important as high concentrations of capital are a defining feature of cities. It is also important to give a role for intermediate inputs of the final manufactured good in production, both because such inputs account for around half of gross output, and because this ensures that productivity is higher in locations where the final good is relatively cheap, generating further agglomerative pressure. We assume that the raw good is untradeable across locations, since it reflects production services.

Firms producing the raw good at location x in period t have profits:

$$\mathcal{P}_{x,t}Z_{x,t} - \mathcal{R}_{K,x,t}K_{x,t-1} - W_{x,t}H_{x,t} - P_{x,t}M_{x,t},$$

and thus choose $K_{x,t-1}$ such that:

$$(1-\kappa)\,\alpha\mathcal{P}_{x,t}\frac{Z_{x,t}}{K_{x,t-1}} = \mathcal{R}_{K,x,t},$$

$H_{x,t}$ such that:

$$(1-\kappa)\,(1-\alpha)\,\mathcal{P}_{x,t}\frac{Z_{x,t}}{H_{x,t}} = W_{x,t},$$

and $M_{x,t}$ such that:

$$\kappa\mathcal{P}_{x,t}\frac{Z_{x,t}}{M_{x,t}} = P_{x,t}.$$

3.6 Households and the representative family

There is a household with population $N_{x,t-1}$ at t at each $x \in X$. Population is pre-determined here to capture the fact that it takes time for people to move to exploit new opportunities elsewhere. As previously mentioned, for simplicity, we assume that all households are part of one representative family that takes decisions on their behalf.

In period t, the family head maximises the discounted utilitarian social welfare function:

$$\mathbb{E}_t\sum_{s=0}^{\infty}\left[\prod_{k=1}^{s}\beta_{t+k-1}\right]\int_X N_{x,t+s-1}\frac{U_{x,t+s}^{1-\varsigma}}{1-\varsigma}\mathrm{d}x,$$

where β_t is the discount factor between periods t and $t+1$, $\varsigma \neq 1$[6] controls risk aversion and intertemporal substitution and:

$$U_{x,t} = \left(\frac{C_{x,t}}{N_{x,t-1}}\right)^{\theta_C}\left(\frac{E_{x,t}}{N_{x,t-1}}\right)^{\theta_F}\left(\frac{1-L_{x,t}}{N_{x,t-1}}\right)^{\theta_L}\left(\frac{1}{1+\nu}\left[1-\left(\frac{H_{x,t}}{N_{x,t-1}}\right)^{1+\nu}\right]\right)^{\theta_H}\cdot$$
$$\left(\Omega\frac{N_{t-1}}{N_{x,t-1}}-1\right)^{\theta_N}g\left(N_{x,t-1},\mathcal{N}_{x,\tilde{x},t}\right), \quad (3.12)$$

where $C_{x,t}$ is total consumption of manufactured goods at x; $E_{x,t}$ the consumption of the agricultural good; $H_{x,t}$ is total labour supplied; and $L_{x,t}$ is access to unfarmed land, capturing the necessity of

[6] The normal device of subtracting 1 from the numerator is not possible here, as it renders the first order condition for population inconsistent with balanced growth.

space for housing, which is otherwise unmodeled.[7] The term weighted by θ_N captures a preference for moderate population density. This is required for model stability; a single productivity shock at x would have otherwise have a permanent effect due to the agglomeration effect. Although stability is required to find a numerical solution, this is otherwise a desirable property, so we set θ_N as low as possible so that there is large endogenous persistence. The last term take the form

$$g\left(N_{x,t-1}, \mathcal{N}_{x,\tilde{x},t}\right) = \left(1 - \frac{\mathcal{N}_{x,t}}{N_{x,t-1}}\right)^{\psi_1} \left(\overline{d} - \frac{\mathcal{D}_{x,t}}{\mathcal{N}_{x,t}}\right)^{\psi_2} \exp\left[\psi_3 \int_X \frac{N_{\tilde{x},t-1}}{N_{t-1}} \log \frac{\mathcal{N}_{x,\tilde{x},t}}{N_{x,t-1}} \, \mathrm{d}\tilde{x}\right] \tag{3.13}$$

where

$$N_{t-1} \equiv \int_X N_{\tilde{x},t-1} \mathrm{d}\tilde{x}, \tag{3.14}$$

$$\mathcal{N}_{x,t} \equiv \int_X \mathcal{N}_{x,\tilde{x},t} \mathrm{d}\tilde{x}, \tag{3.15}$$

$$\mathcal{D}_{x,t} \equiv \int_X d\left(x, \tilde{x}\right) \mathcal{N}_{x,\tilde{x},t} \mathrm{d}\tilde{x}. \tag{3.16}$$

The first term in (3.13) reflects the disutility of having substantial outward migration $\mathcal{N}_{x,t}$, that is, capturing people being upset by their friends and relatives moving away. This ensures that there is always an interior solution for $\mathcal{N}_{x,t}$, which is necessary for tractability. The second term gives the disutility of migration to distant locations, where $\overline{d}$ gives the maximum distance between points, and $\mathcal{D}_{x,t}/\mathcal{N}_{x,t}$ is a measure of the average distance moved. This term helps avoid rapid jumps in population to distant locations, implying that most migration will be between a city and its suburbs. The final term in $g\left(\cdot\right)$ reflects a preference to have at least some migration to all locations ($\mathcal{N}_{x,\tilde{x},t}$ is the amount of migration from location x to location $\tilde{x}$ at t), with higher weight (and so higher migration) to locations with higher populations, i.e. it captures the inevitability of people starting new households with people from far away. This ensures that there is an interior solution for $\mathcal{N}_{x,\tilde{x},t}$, which is necessary for tractability. Finally, the weights are constrained so

$$1 = \theta_C + \theta_F + \theta_L + \theta_H + \theta_N + \psi_1 + \psi_2 + \psi_3. \tag{3.17}$$

The family head faces the budget constraint:

$$\begin{aligned} \int_X \left(P_{x,t} C_{x,t} + E_{x,t}\right) \mathrm{d}x + B_t = \int_X \left(\mathcal{R}_{L,x,t} L_{x,t} + W_{x,t} H_{x,t}\right) \mathrm{d}x + R_{t-1} B_{t-1} \\ - \int_x \Phi_L \left(\frac{L_{x,t}}{L_{x,t-1}}\right) L_{x,t} \mathrm{d}x + T_t, \end{aligned} \tag{3.18}$$

[7] The broad form of the utility function is dictated by the requirement that the model be consistent with balanced growth. This is particularly onerous in this model since the first order condition for population will include $U_{x,t}$, thus we cannot have additive terms within a household's felicity that have different growth rates.

where T_t includes all net profits from owning firms and capital holding companies, and where the family's bond holdings, B_t, are zero in equilibrium.. There is strong evidence for costs of land development due to both construction costs and regulatory costs, and so we include a land-use adjustment cost represented by function Φ_{L}.[8] The family head also faces the following constraint on the evolution of $N_{x,t}$, for all $x \in X$:

$$N_{x,t} = G_{N,t} N_{x,t-1} - \int_X \mathcal{N}_{x,\tilde{x},t} \mathrm{d}\tilde{x} + \int_X \mathcal{N}_{\tilde{x},x,t} \mathrm{d}\tilde{x}, \tag{3.19}$$

where $G_{N,t}$ is the growth rate of aggregate population N_t, $\int_X \mathcal{N}_{x,\tilde{x},t} \mathrm{d}\tilde{x}$ is outwards migration from x and $\int_X \mathcal{N}_{\tilde{x},x,t} \mathrm{d}\tilde{x}$ is inwards migration to x. Writing $\mu_{N,x,t}$ for the Lagrange multiplier on the law of motion for $N_{x,t}$, equation (3.19), we may derive the following first order conditions[9] for consumption $C_{x,t}$:

$$\theta_C E_{x,t} = \theta_F P_{x,t} C_{x,t},$$

land $L_{x,t}$:

$$\theta_L E_{x,t} = \theta_F \begin{pmatrix} \mathcal{R}_{L,x,t} - \Phi_L \left(\frac{L_{x,t}}{L_{x,t-1}} \right) - \Phi'_{\mathrm{L}} \left(\frac{L_{x,t}}{L_{x,t-1}} \right) \frac{L_{x,t}}{L_{x,t-1}} \\ +\beta \mathbb{E}_t \left[\frac{N_{x,t}}{N_{x,t-1}} \frac{E_{x,t+1}}{E_{x,t}} \left(\frac{U_{x,t+1}}{U_{x,t}} \right)^{1-\varsigma} \Phi'_{\mathrm{L}} \left(\frac{L_{x,t+1}}{L_{x,t}} \right) \left(\frac{L_{x,t+1}}{L_{x,t}} \right) \right] \end{pmatrix} (1 - L_{x,t}),$$

hours $H_{x,t}$:

$$\theta_H \left(\frac{H_{x,t}}{N_{x,t-1}} \right)^{\nu} = \theta_F \frac{N_{x,t-1}}{E_{x,t}} W_{x,t} \left(\frac{1}{1+\nu} \Gamma^{1+\nu} - \frac{1}{1+\nu} \left(\frac{H_{x,t}}{N_{x,t-1}} \right)^{1+\nu} \right),$$

bonds B_t:

$$1 = \beta_t R_t \mathbb{E} \frac{N_{x,t} E_{x,t} U_{x,t+1}^{1-\varsigma}}{E_{x,t+1} N_{x,t-1} U_{x,t}^{1-\varsigma}},$$

population $N_{x,t}$:

$$\mu_{N,x,t} = \beta_t \mathbb{E}_t \left[\begin{array}{c} \mu_{N,x,t+1} G_{N,t+1} + U_{x,t+1}^{1-\varsigma} \\ + (1-\varsigma) U_{x,t+1}^{1-\varsigma} \left[\begin{array}{c} \theta_H \frac{\left(\frac{H_{x,t+1}}{N_{x,t}} \right)^{1+\nu}}{\frac{1}{1+\nu} \Gamma^{1+\nu} - \frac{1}{1+\nu} \left(\frac{H_{x,t+1}}{N_{x,t}} \right)^{1+\nu}} - \theta_N \frac{\log\left(\frac{N_{x,t}}{N_t} \right)}{\frac{1}{2}\Omega^2 - \frac{1}{2} \left(\log\left(\frac{N_{x,t}}{N_t} \right) \right)^2} \\ + \psi_1 \frac{\mathcal{N}_{x,t+1}}{N_{x,t} - \mathcal{N}_{x,t+1}} - (\theta_C + \theta_F + \theta_L + \psi_3) \end{array} \right] \end{array} \right],$$

[8] See Glaeser and Gyourko (2018) for a discussion.

[9] This is somewhat complicated by the need to differentiate with respect to functions. We solve this by first replacing expressions of the form $\int_X f(x) \, \mathrm{d}x$ with $\frac{1}{|\tilde{X}|} \sum_{x \in \tilde{X}} f(x)$ where $\tilde{X} \subset X$ is a finite set. We then simplify, and take limits as $\left|\tilde{X}\right| \to \infty$ and as $\tilde{X}$ becomes dense in X. The Lagrangian, and further details on the derivation of these conditions is contained in Appendix C.1

and migration $\mathcal{N}_{x,\tilde{x},t}$:

$$\mu_{N,x,t} = \mu_{N,\tilde{x},t} + (1-\varsigma)\, N_{x,t-1} U_{x,t}^{1-\varsigma} \left[\psi_3 \frac{N_{\tilde{x},t-1}}{N_{t-1}\mathcal{N}_{x,\tilde{x},t}} - \psi_1 \frac{1}{N_{x,t-1} - \mathcal{N}_{x,t}} - \psi_2 \frac{d\,(x,\tilde{x})\,\mathcal{N}_{x,t} - \mathcal{D}_{x,t}}{\bar{d}\mathcal{N}_{x,t}^2 - \mathcal{N}_{x,t}\mathcal{D}_{x,t}} \right].$$

We also have that the representative family's stochastic discount factor is given by:

$$\Xi_{t+1} \equiv \beta_t \frac{N_{x,t} E_{x,t} U_{x,t+1}^{1-\varsigma}}{E_{x,t+1} N_{x,t-1} U_{x,t}^{1-\varsigma}},$$

and that $\frac{E_{x,t}}{N_{x,t-1}U_{x,t}^{1-\varsigma}} = \frac{E_{\tilde{x},t}}{N_{\tilde{x},t-1}U_{\tilde{x},t}^{1-\varsigma}}$ for all $x, \tilde{x} \in X$, implying that with $\varsigma > 1$, households with low utility have high food consumption (a pattern that certainly holds in the data). Note also that with $\varsigma > 1$, $\mu_{N,x,t}$ gives a measure of the undesirability of location x, so it will be optimal to reduce population in locations in which $\mu_{N,x,t}$ is high.

3.7 Market clearing

The final manufactured good is used for consumption $C_{x,t}$, investment $I_{x,t}$ and as an intermediate in raw good production $M_{x,t}$, giving the period t market clearing condition:

$$Y_{x,t} = C_{x,t} + I_{x,t} + M_{x,t}.$$

Since raw goods are used in farming, firm entry and by the producers of differentiated varieties, demand for raw goods in period t is:

$$\begin{aligned} Z_{x,t} &= Z_{F,x,t} + \phi_t \left[J_{x,t} - (1-\delta_J)\, J_{x,t-1}\right] + \int_0^{J_{x,t}} Z_{j,x,t} \mathrm{d}j \\ &= Z_{F,x,t} + \phi_t \left[J_{x,t} - (1-\delta_J)\, J_{x,t-1}\right] + J_{x,t}(1+\lambda)^{-\frac{1+\lambda}{\lambda}} \mathcal{P}_{x,t}^{-\frac{1+\lambda}{\lambda}} \overline{Y}_{x,t}, \end{aligned}$$

where to derive the second line we have used equations (3.4) and (3.6). The agricultural product is only "eaten", and is freely traded across locations, giving the period t market clearing condition:

$$\int_X E_{x,t} \mathrm{d}x = \int_X F_{x,t} \mathrm{d}x.$$

3.8 Stochastic processes

We close the model by specifying the driving stochastic processes. We assume that productivity $A_{x,t}$ is driven by a permanent component that is not location specific, A_t^{P}, and a location specific transitory component, $A^{\mathrm{T}}_{x,t}$. In particular:

$$A_{x,t} = A^{\mathrm{P}}_t A^{\mathrm{T}}_{x,t},$$

where:

$$A^{\mathrm{P}}_t = G_{A,t} A^{\mathrm{P}}_{t-1},$$

and where $\log G_{A,t}$ follows the AR (1) process:

$$\log G_{A,t} = (1 - \rho_{G_A}) \log G_A + \rho_{G_A} \log G_{A,t-1} + \sigma_{G_A} \varepsilon_{G_A,t},$$

and $\log A^{\mathrm{T}}_{x,t}$ follows the spatial AR (1) process:

$$\log A^{\mathrm{T}}_{x,t} = \rho_{A^{\mathrm{T}}} \log A^{\mathrm{T}}_{x,t-1} + \sigma_{A^{\mathrm{T}}} \varepsilon_{A^{\mathrm{T}},x,t},$$

where $\varepsilon_{A^{\mathrm{T}},x,t}$ is a realisation of some continuous stochastic process on X. We also assume that the other aggregate stochastic variables follow AR (1) processes, with:

$$\begin{aligned}
\log G_{N,t} &= (1 - \rho_{G_N}) \log G_N + \rho_{G_N} \log G_{N,t-1} + \sigma_{G_N} \varepsilon_{G_N,t}, \\
\log \tau_t &= (1 - \rho_\tau) \log \tau + \rho_\tau \log \tau_{t-1} + \sigma_\tau \varepsilon_{\tau,t}, \\
\log \phi_t &= (1 - \rho_\phi) \log \phi + \rho_\phi \log \phi_{t-1} + \sigma_\phi \varepsilon_{\phi,t}, \\
\log \beta_t &= (1 - \rho_\beta) \log \beta + \rho_\beta \log \beta_{t-1} + \sigma_\beta \varepsilon_{\beta,t}.
\end{aligned}$$

4 Results

4.1 Growth rates

From combining the model's equilibrium conditions, it may be shown that the model admits a balanced growth path in which for any x, $Y_{x,t}$ has stochastic trend:

$$G_{\tilde{Y},t} \equiv (G_{A,t} G_{N,t})^{\frac{(1-\alpha)(1-\kappa)(1+\lambda)}{(1-\alpha)(1-\kappa)(1+\lambda)-\lambda}}.$$

(Note, this does not mean that $G_{\tilde{Y},t}$ will equal $\frac{Y_{x,t}}{Y_{x,t-1}}$ for any particular x. Rather, this means that $\frac{Y_{x,t}}{\tilde{Y}_t}$ will be stationary, where $\tilde{Y}_t$ evolves according to $\tilde{Y}_t = G_{\tilde{Y},t} \tilde{Y}_{t-1}$.) Since $(1-\alpha)(1-\kappa)(1+\lambda) - \lambda > 0$ in any reasonable calibration, this implies that the growth rate of output is higher than that of $G_{A,t} G_{N,t}$.

Thus, this is a model of semi-endogenous growth. Exogenous growth in productivity or population leads to further endogenous growth since it increases the measure of firms producing differentiated varieties, which feeds into the love for variety embedded in our aggregator, equation (3.1). The smaller is λ, the weaker will be this endogenous growth channel, with purely exogenous growth in the $\lambda = 0$, perfect competition, limit. The presence of this channel is also suggestive of areas of high population ("cities") having higher productivity.

The stochastic trend of other variables may be given in terms of the stochastic trend in output. In particular, we have that for any x, the stochastic trend in $Z_{x,t}$ and $J_{x,t}$ is given by $G_{\tilde{Y},t}^{\frac{1}{1+\lambda}}$, the stochastic trend in $F_{x,t}$, $E_{x,t}$ and $\mathcal{R}_{L,x,t}$ is given by $G_{\tilde{Y},t}^{\frac{1-\gamma}{1+\lambda}}$, the stochastic trend in $W_{x,t}$ is given by $G_{\tilde{Y},t}^{\frac{1-\gamma}{1+\lambda}} G_{N,t}^{-1}$, the stochastic trend in $P_{x,t}$, $Q_{x,t}$ and $\mathcal{R}_{K,x,t}$ is given by $G_{\tilde{Y},t}^{-\frac{\gamma+\lambda}{1+\lambda}}$, the stochastic trend in $\mathcal{P}_{x,t}$ and $\mathrm{D}_{x,t}$ is given by $G_{\tilde{Y},t}^{-\frac{\gamma}{1+\lambda}}$, the stochastic trend in $\overline{Y}_{x,t}$ is given by $G_{\tilde{Y},t}^{-\frac{\gamma}{\lambda}}$, the stochastic trend in $U_{x,t}$ is given by $G_{\tilde{Y},t}^{\theta_C+\theta_F\frac{1-\gamma}{1+\lambda}} G_{N,t}^{-(\theta_C+\theta_F+\theta_L)}$, and the stochastic trend in $\mu_{N,x,t}$ is given by $G_{\tilde{Y},t}^{\left(\theta_C+\theta_F\frac{1-\gamma}{1+\lambda}\right)(1-\varsigma)} G_{N,t}^{-(\theta_C+\theta_F+\theta_L)(1-\varsigma)}$. Thus, amongst other results, the model predicts that the price of manufactured goods and capital is falling with respect to the price of agricultural goods, and that consumption of agricultural goods is growing less quickly than consumption of manufactured ones.[10]

4.2 Properties of the steady-state, and choice of space and spatial correlation

While the full steady-state of the detrended model does not admit a closed form solution, in the special case in which the space X is invariant under translation (i.e. X is a circle or a torus), then the detrended model admits a uniform steady-state in which all variables are constant over x, and in which some variables have a closed form solution. In particular, in the uniform solution in the absence of shocks, $L_{x,t} = \frac{\gamma\theta_F}{\theta_L+\gamma\theta_F}$ and $\frac{\mathcal{N}_{x,t}}{N_{t-1}} = \frac{\psi_3}{\psi_1+\psi_3}$. Thus, the steady-state amount of land used in agriculture is increasing in the importance of land for agricultural production, and in the importance of food for utility, and decreasing in the importance of land for utility. Additionally, the steady-state amount of migration is increasing in the family's desire to have at least some migration to each location, and decreasing in the amount the family dislikes any migration.

When the space X is not invariant under translation, as in the case when X is the plane $[0,1] \times [0,1]$ with the usual Euclidean metric, numerical results suggest that the steady-state features a significant concentration of population around the centre, $\left(\frac{1}{2}, \frac{1}{2}\right)$. To see why this is unsurprising, suppose that population were initially uniformly distributed. Then the centre would have lower average transport costs, since it is on average closer to other places. These lower transport costs would imply lower prices and higher productivity in the centre, making it an attractive destination for migration.

[10] The multiple different stochastic trends in the model complicate its simulation. However, this is facilitated by the ability of the toolkit we provide here: `https://github.com/tholden/DynareTransformationEngine` to automatically take care of detrending variables, once the stochastic trends are supplied.

In reality, in the U.S. at least, we see a lot of population on the coasts, and less in the centre. This is partly down to historical artefact, as the coasts were settled first, and partly due to the fact that in reality the coasts have low transport costs both to other points on the coast, and to the rest of the world. Rather than modelling trade along the coast, and to the rest of the world, we keep things simple by modelling space as a torus, identified with $[0,1] \times [0,1]$. That is, our model of space is similar to how the continental U.S. would be were it the case that if you crossed the Canadian border, you teleported onto an equivalent point on the Mexican border, and if you stepped off the pier in Boston, you teleported to Seattle. We place the usual Euclidean norm on the torus, i.e.:

$$d\left([x_1, x_2], [\tilde{x}_1, \tilde{x}_2]\right) = \sqrt{\left(\min\left\{\left|x_1 - \tilde{x}_1\right|, 1 - \left|x_1 - \tilde{x}_1\right|\right\}\ \right)^2 + \left(\min\left\{\left|x_2 - \tilde{x}_2\right|, 1 - \left|x_2 - \tilde{x}_2\right|\right\}\ \right)^2}.$$

Given this, it makes sense to use the "natural" continuous stochastic process on the torus introduced in Section 2.1, so:

$$\operatorname{cov}\left(\varepsilon_{A^{\mathrm{T}},x,t}, \varepsilon_{A^{\mathrm{T}},\tilde{x},t}\right) \ = s\left(\zeta, d\left(x, \tilde{x}\right)\right),$$

where s is as defined in Section 2.1.

4.3 Calibration

We now describe the calibration used for the simulation exercises. Some of the parameters are calibrated to target some steady state ratios, values of which are shown in the top part of table 1 with further detail outlined in appendix B. The last three parameters in the table are calibrated to target higher order simulated moments. In this section, we discuss the parameters which affect the stability properties of the model and control population dynamics, namely: $\{\theta_N, \Omega, \varrho, \psi_3, \zeta\}$. We calibrate ζ jointly with the intertemporal persistence of the spatial productivity shock, $\rho_{A^{\mathrm{T}}}$, by using state-level productivity data to estimate

$$\log A_{i,t} = \alpha_t + \beta_i + \gamma_i t + \sigma_i \varepsilon_{i,t} \tag{4.1}$$

Capturing state-level trends, and time and state fixed effects, where for all $i, j \in \{1, \cdots, N\}$ and $t, s \in \{1, \cdots, T\}$:

$$\operatorname{cov}\left(\varepsilon_{i,t}, \varepsilon_{j,s}\right) = \rho_{A^{\mathrm{T}}}^{|t-s|} \exp\left[-\zeta d_{i,j}\right] \tag{4.2}$$

Parameter	Value	Target
G^A	1.0026	Quarterly trend growth rate in output $G^Y = 0.679\%$
G^N	1.0026	Quarterly trend growth rate in population $= 0.262\%$
β	0.997	Real interest rate: $R = 1.0056$
α	0.18	Ratio of consumption to investment: $\frac{P_x C_x + E_x}{P_x I_x} = 4.80$
κ	0.50	Ratio of manufactured goods cost to total value of production: $\frac{P_{x,t} M_{x,t}}{\mathcal{P}_{x,t} Z_{x,t}} = \kappa = 0.5$
γ	0.18	Income share of land: $\frac{\mathcal{R}_{L,x,t} L_{x,t}}{F_{x,t}} = \gamma = 0.18$
λ	0.1	Median markup
δ_J	0.0268	Quarterly establishment exit rate, $\delta_J = 2.682\%$
δ_K	0.0136	Average quarterly capital depreciation rate, $\delta = 1.36\%$
θ_C	0.144	Utility weights sum to 1, $1 = \theta_C + \theta_F + \theta_L + \theta_H + \theta_N + \psi_1 + \psi_2 + \psi_3$
θ_F	0.015	Food as a proportion in consumption bundle $\frac{\int_X E_{x,t} \mathrm{d}x}{\int_X P_{x,t} C_{x,t} + E_{x,t} \mathrm{d}x} = 9.485\% \Longrightarrow \theta_F = \theta_C \frac{0.09485}{1-0.9485}$
θ_H	0.687	Average hours, $\int_X H_{x,t} \mathrm{d}x = 0.4109$
θ_L	0.0005	Proportion of land used for agriculture, $L_{x,t} = 0.84 \Longrightarrow \theta_L = \frac{\gamma}{3} \theta_F$
ψ_1	0.0013	Proportion of households that move each quarter, $\int_X \mathcal{N}_{x,t} / N_{x,t} \mathrm{d}x = \frac{1}{13 \times 4} \Longrightarrow \psi_3 = \frac{0.01923}{\varrho - 0.01923} \psi_1$
ψ_2	0.0001	40% of moves are over distances of more than 100 miles. i.e., $d = 0.0155$ and 23% of moves are over distances of more than 500 miles. i.e., $d = 0.0773$
ν	2	Standard deviation of total hours, $\int_X H_{x,t} \mathrm{d}x$, 3.89%
ς	1.5	Relative standard deviation of consumption, $\int_X P_{x,t} C_{x,t} + E_{x,t} \mathrm{d}x$, 0.7685
Φ	4	Relative standard deviation of investment, $\int_X P_{x,t} I_{x,t} \mathrm{d}x$, 4.5858

Table 1: Calibrations and targets

where $\varepsilon_{i,t}$ is normally distributed with $\mathbb{E}\varepsilon_{i,t} = 0$.[11] This resulted in $\zeta = 14.16$[12] and $\rho_{A^{\mathrm{T}}} = 0.763$. The parameter $\bar{d}$ is implied by the dimension and type of space. With a 1×1 torus, the maximum possible distance between two points is $0.5\sqrt{2}$.

4.4 Impulse responses

To understand the dynamic behaviour of our model, we start by simulating impulse responses.[13] In all of the simulations reported here we used a grid with effective size 9×9, with the bottom row of grid points always agreeing with the top row, and the right column always agreeing with the left column.

We start by looking at the effects of a 3.6% spatial productivity shock. (that is, a magnitude 1 standard deviation shock.) Since space is invariant under translation, without loss of generality we may focus

[11]The model was estimated via maximum likelihood, profiling out all parameters except $\rho_{A^{\mathrm{T}}}$ and ζ from the likelihood. Conditional on $\rho_{A^{\mathrm{T}}}$ and ζ, the other parameters were estimated by iterated feasible generalized least squares, with iteration until convergence of σ. The estimate of σ used within each iteration was the maximum likelihood one, which satisfies $\sigma = \frac{1}{T} \mathrm{diag} \left[E P^{-1} E^{\mathrm{T}} (\mathrm{diag} \sigma) Z^{-1} \right]$, where the diag operator maps matrices to vectors containing their diagonal, and vectors to matrices with zeros apart from the given diagonal, where $E = [\varepsilon_{i,t}]_{\substack{t=1,\cdots,T \\ i=1,\cdots,N}}$, $P = \left[\rho_{A^{\mathrm{T}}}^{|t-s|} \right]_{\substack{t=1,\cdots,T \\ s=1,\cdots,T}}$ and $Z = [\exp[-\zeta d_{i,j}]]_{\substack{i=1,\cdots,N \\ j=1,\cdots,N}}$.

[12]With a 95% confidence interval $[11.81, 17.12]$, produced by inverting the likelihood ratio test on the profile likelihood for ζ.

[13]The code we used both to simulate impulse responses, and to simulate stochastic runs is available from: `https://github.com/tholden/DynamicSpatialModel`. This repository also includes the full set of these results, including videos showing the evolution over time of all distributions.

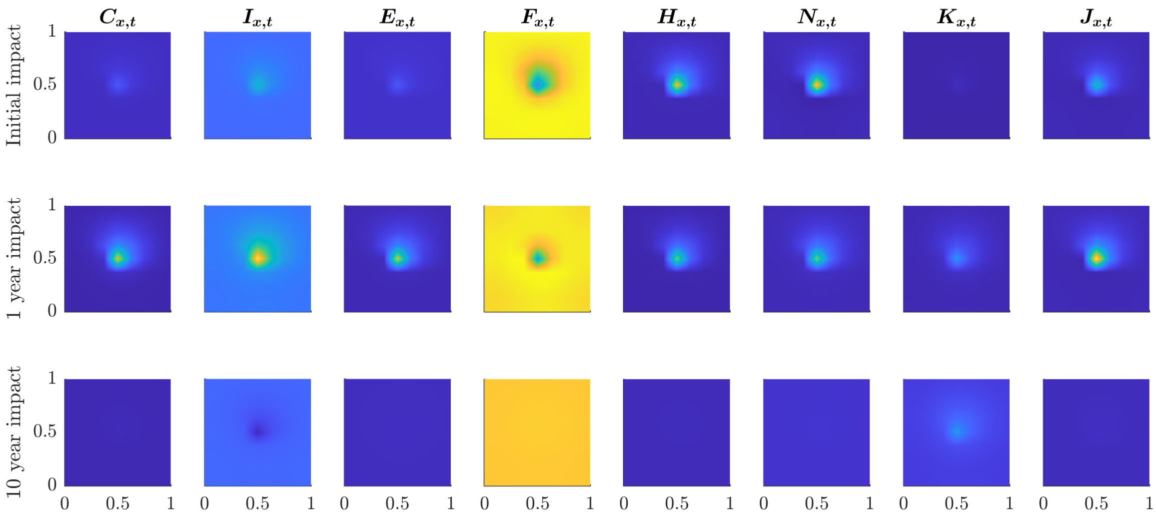

Figure 3: Impulse responses of key state variables to a 1% spatial productivity shock centred on $\left(\frac{1}{2}, \frac{1}{2}\right)$. Entire space, snapshots in time. Bright colours are high values.

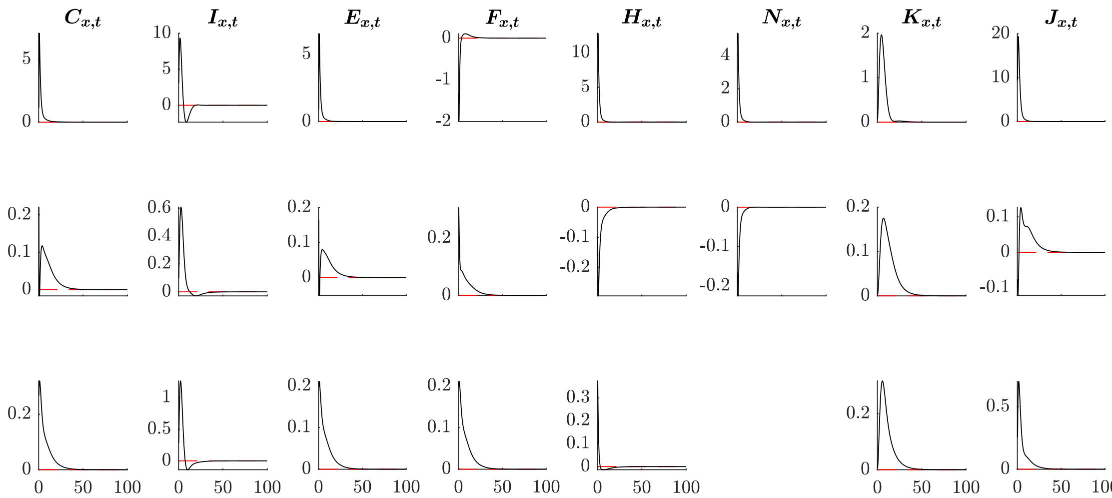

Figure 4: Impulse responses of key state variables to a 1% spatial productivity shock centred on $\left(\frac{1}{2}, \frac{1}{2}\right)$. Percent deviation from steady state. x-axis measured in years.

on a shock that is centred on the point $\left(\frac{1}{2}, \frac{1}{2}\right)$. As shocks are correlated across locations, we take the matrix square root of the covariance matrix to determine the impulse at each location. The impact of such a shock is shown in Figures 3 and 4. Note that where an aggregate IRF is shown, this gives the IRF to the integral of the variable over X. In the density plots in , bright colours represent high values. The shock leads to an increase in consumption (manufactured and agricultural), investment, capital, the measure of firms, hours and population at the epicentre. Despite this, utility at the epicentre actually falls as people there are asked to work harder to take advantage of their high productivity. Since the increase in productivity leads to firm entry, it is optimal for the epicentre to move away from agricultural production, towards manufacturing. Consequently, agricultural production increases elsewhere, with a consequent increase in the land used for farming. This does not harm utility away from the epicentre since population flows towards the centre, reducing pressure on land in the periphery. As one would expect, aggregate utility increases overall from this positive productivity shock.

The effects of this initial shock are extremely persistent, with population still not back to trend 100 years after the initial shock. This suggests that our model can successful explain how small initial shocks can lead to city formation in one place, and not in another.

One explanation for the decline in the U.S. midwest through the lens of the model is that the increase in productivity in e.g. San Francisco and New York has pulled people out of the midwest and towards the coasts. That utility seems to have declined in the midwest suggests that our model is still lacking important frictions, such as costs to adjust land usage.

5 Conclusion

This paper has presented a new approach to building heterogeneous agent macroeconomic models in which the heterogeneity is across space. While the paper focuses on applications in which space is physical space, our approach can also contribute to understanding heterogeneity across types, be it product variety, skills or preferences.

We suggested that spatial macroeconomic models should be driven by shocks that are continuous across space, and presented a variety of examples of such shock processes. We give further technical results on existence of such processes across a wide range of spaces of interest in Appendix A.

We went on to build a DSGE model featuring the key model components of the new economic geography literature. We showed that the model was able to generate extremely persistent movements in population, even given very strong preferences for a moderate population density. Thus, this is a model in which business cycle shocks can endogenously lead to the formation of new cities.

In future work, we plan on extending the model presented here, incorporating, for example, adjustment costs to land, that might ameliorate the need to have a preference for moderate population density. We will also undertake a more comprehensive calibration exercise, explore the asymmetric steady-states of the model, and assess the feasibility of solving the model at a higher order of approximation to capture the model's important non-linearities.

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Online Appendices for "Business Cycles in Space"

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Appendix A Continuous stochastic processes

A.1 Type spaces

We would like to be able to draw realisations of a shock which are continuous (and hence locally correlated) over some compact space X, equipped with a Radon measure. Compactness of X ensures that the law of large numbers does not remove the aggregate impact of the stochastic process, and the existence of a Radon measure is a technical assumption that will not rule out any spaces of interest.

Often, we will have $X = h^{-1}(G)$, where $h : X \to G$ is continuous, and G is a locally-compact, abelian group, also equipped with some Radon measure. For those readers not familiar with group theory, one may view a group as a structure on which "addition" and "subtraction" are defined, along with an identity "zero". The addition and subtraction operations can be thought of as spatial translations. This "additive" group structure will be important, since it will give rise to the spatial analogue of the time series procedure of taking lags or leads.

We will define an underlying continuous stochastic process on G, a realisation of which will be a continu-

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ous function $\phi : G \to \mathbb{R}$. It will turn out that continuous stochastic processes will be particularly easy to construct on such groups, explaining our interest in type spaces (Xs), that admit such a representation. The realisation of the stochastic process on X will then be given by $x \mapsto \phi(h(x))$, which is continuous by the continuity of ϕ and h. If not stated otherwise, h will be the identity map (or more strictly, the inclusion map). The assumption that G is abelian (i.e. the group operation is commutative) is not needed, but all practical examples will feature abelian G, so nothing is lost. Throughout, the group operation will be denoted "+", with inverse "−" and identity element "0".

For example, we might have:

- $X = [0,1]^n \subseteq \mathbb{R}^n = G$, for some $n \in \mathbb{N}$. This might represent a space of types in which there is a meaningful boundary, such as education levels, or with $n = 2$, the physical area of a country. The group operation and measure are the normal ones on $\mathbb{R}^n$.
- $X = \mathbb{S}_n$, for some $n \in \mathbb{N}$, where $\mathbb{S}_n = \left\{x \in \mathbb{R}^{n+1} \,\middle|\, \|x\| = 1\right\}$ (i.e. circle, sphere, etc.). This might represent a space of types with no meaningful boundary, in which there is no intrinsic difference between the axes of the type space, or with $n = 2$, it can represent the surface of earth (a sphere!). The group operation on $\mathbb{S}_1$ is addition of angles, so in that case we may take $G = X_1$. Spheres in three-dimensional space do not have an Abelian group operation, so need to be treated specially. The measure on the space is given by the usual spherical surface element.
- $X = \mathbb{T}_n = G$, for some $n \in \mathbb{N}$, where $\mathbb{T}_n = \mathbb{R}^n/\mathbb{Z}^n \simeq \mathbb{S}_1^n$ (i.e. circle, torus, etc.). This gives another representation for type spaces with no boundary, this time for type spaces with clear axes/anisotropy. In this case, the group operation and measure are the ones induced by the quotient construction.
- X is the embedding of a graph in $\mathbb{R}^n$ (i.e. a collection of joined curved line-segments). This might represent a road, river, train or canal network. There is no natural group operation in general, so the existence of continuous processes on this space will be non-trivial. The measure in this case is the one dimensional Hausdorff measure in X.

A.2 General results on the existence of continuous stochastic processes

Before introducing our general results on the existence of continuous stochastic processes in spaces such as these, we first need to define two closely related notions of positive definiteness for functions.

Definition 1 *Let $f : G \to \mathbb{R}$. We say f is* ***positive definite on G*** *if f is a positive, even function, and $\forall n \in \mathbb{N}$, $\forall x_1, \ldots, x_n \in G$, the matrix $[f(x_i - x_j)]_{i,j=1,\ldots,n}$ is positive semi-definite.*

Definition 2 *Let $f : X \times X \to \mathbb{R}$. We say f is* ***positive definite on X×X*** *if f is a positive, symmetric* *function, and* $\forall n \in \mathbb{N}$, $\forall x_1, \ldots, x_n \in G$, *the matrix* $[f(x_i, x_j)]_{i,j=1,\ldots,n}$ *is positive semi-definite.*

The two notions are related, as if f is a positive definite function on G, then $(x_1, x_2) \mapsto f(h(x_1) - h(x_2))$ is positive definite on $X \times X$, providing h is as defined previously.

It is easy to verify that sums and products of positive definite functions are positive definite, and that a positive multiple of a positive definite function is positive definite. A further useful characterisation of positive definite functions is Bochner's theorem, which, in our context, implies that f is positive definite on G if and only if the Fourier transform (equivalently, cosine transform) of f on G is positive. For spaces with a group structure, this gives an easy method of constructing positive definite functions.

We are interested in positive definite functions, as by results due to Doob, for the existence of a Gaussian stochastic process with covariance $f(x_i, x_j)$ for all $x_1, x_2 \in X$, it is sufficient that the function f be positive definite on $X \times X$. Hence, by using this result along with Bochner's theorem, we are easily able to verify if there is a continuous stochastic process with the covariance structure we desire. Alternatively, if we find results in the mathematical literature proving the positive definiteness of some function of interest, then we know there is a continuous stochastic process with that auto-covariance function.

A.3 Continuous stochastic processes in Euclidean spaces

We now proceed to give examples of spaces and some of the possible continuous stochastic processes on those spaces. Ornstein-Uhlenbeck processes extend naturally to Euclidean spaces, and may be further generalized to allow for different rates of decay of the auto-covariance. In particular, let $\|\|_p$ be the usual ℓ_p (quasi-)norm on $\mathbb{R}^n$, then $x \mapsto \exp\left\{-\|x\|_p^q\right\}$ is positive definite on $\mathbb{R}^n$ if and only if one of the following conditions hold:

- $0 < q \leq p \leq 2$ (independent of n).
- $n = 1$, and $p > 0$, and $0 < q \leq 2$.
- $n = 2$, and $p \in (2, \infty]$, and $q \in (0, 1]$,

(Schoenberg, 1938; Misiewicz, 1989; Koldobsky, 1992; Kuniewski and Misiewicz, 2014). This gives us a wide range of stochastic processes on $\mathbb{R}^n$, with the $p = 1$, $q = 1$ and $p = 2$, $q = 1$ processes both being contenders to be the "natural" Ornstein-Uhlenbeck process on $\mathbb{R}^n$. The $p = q = 2$ process is also potentially useful in macroeconomic applications, as it is the unique stochastic process in this class with realisations that are smooth almost surely.

Another useful class of positive definite functions on $\mathbb{R}^n$ are given by $x \mapsto (1 + \|x\|_2^\alpha)^{-\frac{\tau}{\alpha}}$, where $\alpha \in (0, 2]$ and $\tau > 0$ (Gneiting and Schlather, 2004). These have long-memory, so may be useful in applications with very high spatial dependence. Further processes on $\mathbb{R}^n$ may be constructed by taking $G = \mathbb{R}^m$, $X = \mathbb{R}^n$, and $h(x) = Ax$ for some matrix A, then $f(h(x_1) - h(x_2)) = f(h(x_1 - x_2))$ so $x \mapsto \exp\left\{-\|Ax\|_p^q\right\}$ and $x \mapsto (1 + \|Ax\|_2^\alpha)^{-\frac{\tau}{\alpha}}$ are positive definite with the same assumptions as before.

A.4 Continuous stochastic processes on circles and spheres

If $X = \mathbb{S}_n$, the sphere in $n+1$ dimensional space, then the natural distance between points is the great circle distance, which, appropriately normalised, is given by $d(x_1, x_2) = \frac{1}{2\pi}\arccos(x_1' x_2) \in \left[0, \frac{1}{2}\right]$. Then, for $\zeta > 0$, $(x_1, x_2) \mapsto \exp\{-\zeta d(x_1, x_2)^q\}$ is positive definite on $X \times X$ if and only if $q \in (0, 1]$ (Bogomolny et al., 2007). On $\mathbb{S}_1$, this function has cosine transform $k \mapsto \frac{\zeta\left(1-(-1)^k e^{-\zeta/2}\right)}{4\pi^2 k^2 + \zeta^2}$ for $k \in \mathbb{N}^+$, which has an undesirable oscillating component not present in the cosine transform on $\mathbb{R}$ of $x \mapsto \exp(-\zeta |x|)$, i.e. $\omega \mapsto \frac{\zeta}{4\pi^2\omega^2 + \zeta^2}$. As a result, this may not be a particularly natural choice. As an alternative, it is worth noting that on $\mathbb{S}_1$,[1] for $\zeta > 0$, the function $(x_1, x_2) \mapsto s(\zeta, d(x_1, x_2))$ is positive definite, where $s(\zeta, d) = \frac{\exp\left(-\zeta d + \frac{\zeta}{2}\right) + exp\left(\zeta d - \frac{\zeta}{2}\right)}{\exp\left(\frac{\zeta}{2}\right) + \exp\left(-\frac{\zeta}{2}\right)}$ (Pedersen, 2002). Furthermore, this has a cosine transform proportional to $k \mapsto \frac{\zeta}{4\pi^2 k^2 + \zeta^2}$ for $k \in \mathbb{N}^+$, which means it is the natural generalisation of the Ornstein-Uhlenbeck process on $\mathbb{R}$. Other possibilities include $(x_1, x_2) \mapsto (1-\zeta) + \zeta\left(\frac{1}{2} - d(x_1, x_2)\right)^2$ and $(x_1, x_2) \mapsto (1-\zeta) + \frac{1}{2}\zeta(1 + x_1' x_2)$, which are both positive definite for $\zeta \in [0, 1]$, by the condition given in Gneiting (2013). The latter is an analogue of $x \mapsto \exp\left(-\zeta x^2\right)$ on $\mathbb{R}$, and will lead to smooth sample paths.

A.5 Continuous stochastic processes on tori

If $G = G_1 \times G_2 \times \cdots \times G_n$, and f_i is positive definite on G_i for $i = 1, \ldots, n$, then the function $f : G \to \mathbb{R}$ defined by $f(x) = \prod_{i=1}^n f_i(x_i)$ is positive definite on G. Hence, positive definite functions on tori can be constructed from products of positive definite functions on circles (where the circle is identified with $\mathbb{R}/\mathbb{Z}$). For example, by our previous results, for $\zeta_1, \ldots, \zeta_n > 0$, $q_1, \ldots, q_n \in (0, 1]$, the function:

$$(x_1, \ldots, x_n) \mapsto \exp\left\{-\sum_{i=1}^n \zeta_i \min\{|x_i|, 1 - |x_i|\}^{q_i}\right\}$$

will be positive definite on G.

A.6 Continuous stochastic processes on graphs or networks

In general, graphs cannot be isometrically embedded in Euclidean space, so if d is the shortest path metric on the (embedding of the) graph, $(x_1, x_2) \mapsto \exp\left(-\zeta d(x_1, x_2)^2\right)$ will only be positive definite for

[1] This does not hold on $\mathbb{S}_2$ or higher, by the condition given in Gneiting (2013).

very particular graphs. We do know however that $(x_1, x_2) \mapsto \exp(-\zeta d(x_1, x_2))$ will be positive definite for all $\zeta > 0$ if the following conditions are all satisfied (Chepoi et al., 1997):

- the graph is unweighted, or possesses integer weights,
- the graph is planar,
- every interior face of the graph is an isometric cycle,
- two interior faces meet at at most one edge (of length one).

Additionally, $(x_1, x_2) \mapsto \exp(-\zeta d(x_1, x_2))$ is positive definite on all (weighted) trees, for all $\zeta > 0$ (Hjorth et al., 1998).

Furthermore, it may be shown that for all graphs, there exists $\overline{\zeta}$ such that $(x_1, x_2) \mapsto \exp(-\zeta d(x_1, x_2))$ is positive definite for $\zeta > \overline{\zeta}$. The idea of the proof is the following. For large ζ, the "ACF" matrix is diagonally dominant on the graph's vertices, hence positive definite, hence we can draw from a finite dimensional Gaussian process on the vertices, and then link the realisations at the vertices with independent Ornstein-Uhlenbeck processes along each, conditional on them taking the given values at the vertices.

Appendix B Calibration details

The exogenous technical progress is chose to target the growth rate in output, calculated by regressing logged real GDP (from NIPA) against time. Population growth is that of the total working age population of the U.S. from the OECD. The real interest comes from applying the Fisher equation $r_t = r_t^N - \mathbb{E}_t \pi_{t+1}$ using the 3-month treasury bill secondary market rate as the risk-free nominal rate and the realised next period GDP deflator from 1980–2010. This yields an average real annual rate of 2.27%. The factor shares are taken as the averages from the BLS 'Multifactor Productivity Trends in Manufacturing' release. We sum over energy, materials and purchased services for the manufactured good inputs and take averages over the whole empirical sample. The income share of land of 0.18 comes from that estimated by Valentinyi and Herrendorf (2008). The price markup is the same used by Comin and Gertler (2006), originally estimated by Basu and Fernald (1997). Establishment exit rate is the average over 1977-2014 according to the Longitudinal Business Database, U.S. Census. The capital depreciation rate is computed by dividing BEA's depreciation of fixed assets by the stock using data over 1980-2018. For θ_L, we use U.S. evidence that indicates the share of land with broadly agricultural usage is around 84%.[2]. For θ_F, using BEA consumption data by category, we find food has by 9.5% of consumption[3] so set

[2]Data from `https://www.ers.usda.gov/data-products/major-land-uses/`, Summary Table 1. We classify cropland, grassland, pasture, range and forest-use land as agricultural, and the remainder as non-agricultural.

[3]Excluding housing from total consumption and excluding food services from food in-line with methodology of BEA.

$\theta_F = \theta_C \frac{0.09485}{1-0.9485}$. For the distribution of distance moved, we use survey data on home-searches that suggests 23% of moves are over a distance of more than 500 miles and 60% less than 100 miles.[4]. These targets are used to pin down ψ_2 The steady state conditions for migration are linear in ψ_1, ψ_2 and ψ_3. The ratio ψ_1/ψ_3 is pinned down by the number of movers, then conditional on any value of ψ_2, we can solve ψ_3 to minimize the sum of squared residuals from the two targets. The maximum distance between counties in the contiguous U.S. is 4572.6 miles, so with a 1×1 torus where $\bar{d} = 0.5\sqrt{2}$, this is 23% moving across distances of more than 0.0773. Similarly, 40% of moves are over distances of more than 100 miles, which is 0.0155 on the 1×1 torus. The proportion of household movers is taken from a study by the National Association of Home Builders.[5] The proportion of households moving in a single quarter is then $\frac{1}{13\times 4} = 0.01923$ and so $\psi_3 = \frac{0.01923}{\varrho - 0.01923}\psi_1$ Hours is the total annual hours worked by FT and PT workers in the U.S. (BEA) divided by working age population of the U.S. (OECD). For the first moment, we divide by 52*7*8.[6] The average hours is only GDP, consumption and investment data is from the BEA's National Income and Product Accounts (NIPA). Consumption is disaggregated to food purchases and food services. Relative standard deviation is the ratio with the standard deviation of total GDP which is taken as the sum of manufactured goods and agricultural goods, $\int_X P_{x,t}Y_{x,t} + F_{x,t}\mathrm{d}x$.

Appendix C Further model properties

C.1 Lagrangians

The capital holding company's problem leads to the following Lagrangian:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left[\prod_{k=1}^{s} \Xi_{t+k} \right] \begin{pmatrix} \mathcal{R}_{K,x,t+s} K_{x,t+s-1} - P_{x,t} I_{x,t+s} + \\ Q_{x,t} \left(\left(1 - \delta_K\right) K_{x,t-1} + \left[1 - \Phi\left(\frac{I_{x,t}}{I_{x,t-1}} \right) \right] I_{x,t} - K_{x,t} \right) \end{pmatrix}.$$

The household's problem leads to the following Lagrangian:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left[\prod_{k=1}^{s} \beta_{t+k-1} \right].$$

[4] https://www.trulia.com/research/moving-far-far-away-or-not-far-enough/.

[5] See http://eyeonhousing.org/2013/01/latest-study-shows-average-buyer-expected-to-stay-in-a-home-13-years/.

[6] This is only used to normalize hours and implies a maximum daily average hours of 8, accounting for holiday time and weekends. If one assumes workers take 4 weeks vacation a year and work 5 day weeks, this is a maximum 12 hours per working day. Conditional on the these assumptions, the average is under 5 hours per working day.

$$
\left[\begin{array}{c}
\int_X \left[\begin{array}{c}
N_{x,t+s-1}\left[\begin{array}{c}
\frac{U_{x,t+s}^{1-\varsigma}}{1-\varsigma}+\mu_{U,x,t+s}\left[\begin{array}{c}
\left(\frac{C_{x,t+s}}{N_{x,t+s-1}}\right)^{\theta_C}\left(\frac{E_{x,t+s}}{N_{x,t+s-1}}\right)^{\theta_F}\left(\frac{1-L_{x,t+s}}{N_{x,t+s-1}}\right)^{\theta_L}\cdot \\
\left(\frac{1}{1+\nu}\Gamma^{1+\nu}-\frac{1}{1+\nu}\left(\frac{H_{x,t+s}}{N_{x,t+s-1}}\right)^{1+\nu}\right)^{\theta_H}\cdot \\
\left(\frac{1}{2}\Omega^2-\frac{1}{2}\left(\log\left(\frac{N_{x,t+s-1}}{N_{t+s-1}}\right)\right)^2\right)^{\theta_N}\cdot \\
\left(1-\frac{\int_X \mathcal{N}_{x,\tilde{x},t+s}\mathrm{d}\tilde{x}}{N_{x,t+s-1}}\right)^{\psi_1}\left(\overline{d}-\frac{\int_X d(x,\tilde{x})\mathcal{N}_{x,\tilde{x},t+s}\mathrm{d}\tilde{x}}{\int_X \mathcal{N}_{x,\tilde{x},t+s}\mathrm{d}\tilde{x}}\right)^{\psi_2}\cdot \\
\exp\left[\psi_3\frac{\int_X N_{\tilde{x},t+s-1}\log\frac{\mathcal{N}_{x,\tilde{x},t+s}}{N_{x,t+s-1}}\,\mathrm{d}\tilde{x}}{\int_X N_{\tilde{x},t+s-1}\mathrm{d}\tilde{x}}\right]-U_{x,t+s}
\end{array}\right]
\end{array}\right] \\
+\frac{\mu_{N,x,t+s}}{1-\varsigma}\left[G_{N,t+s}N_{x,t+s-1}-\int_X \mathcal{N}_{x,\tilde{x},t+s}\mathrm{d}\tilde{x}+\int_X \mathcal{N}_{\tilde{x},x,t+s}\mathrm{d}\tilde{x}-N_{x,t+s}\right]
\end{array}\right]\mathrm{d}x \\
+\mu_{B,t+s}\left[\begin{array}{c}
\int_X\left(\mathcal{R}_{L,x,t+s}L_{x,t+s}+W_{x,t+s}H_{x,t+s}\right)\mathrm{d}x+R_{t+s-1}B_{t+s-1}+T_{t+s} \\
-\int_X\left(P_{x,t+s}C_{x,t+s}+E_{x,t+s}\right)\mathrm{d}x-\int_x \Phi_{\mathrm{L}}\left(\frac{L_{x,t+s}}{L_{x,t+s-1}}\right)L_{x,t+s}dx-B_{t+s}
\end{array}\right]
\end{array}\right].
$$

Note that $\mu_{U,x,t}$ and $\mu_{B,t}$ do not occur in the first order conditions given in the text, as we substitute it out from the first order condition for $U_{x,t}$:

$$\mu_{U,x,t}=U_{x,t}^{-\varsigma}.$$

and the first order condition of $E_{x,t}$:

$$\mu_{B,t}=\theta_F\frac{N_{x,t-1}}{E_{x,t}}U_{x,t}^{1-\varsigma}.$$

C.2 Equilibrium conditions

The complete set of equilibrium conditions of the model are as follows:

$$\overline{Y}_{x,t}\equiv\int_X Y_{\tilde{x},t}P_{\tilde{x},t}^{\frac{1+\lambda}{\lambda}}\exp\left[-\frac{\tau_t}{\lambda}d\left(\tilde{x},x\right)\right]\,\mathrm{d}\tilde{x} \tag{C.1}$$

$$\mathrm{D}_{x,t}\equiv\frac{\lambda}{1+\lambda}(1+\lambda)^{-\frac{1}{\lambda}}\mathcal{P}_{x,t}^{-\frac{1}{\lambda}}\overline{Y}_{x,t} \tag{C.2}$$

$$P_{x,t}=(1+\lambda)\left[\int_X J_{\tilde{x},t}(\mathcal{P}_{x,t}\exp\left[\tau_t d\left(x,\tilde{x}\right)\right])^{-\frac{1}{\lambda}}\mathrm{d}\tilde{x}\right]^{-\lambda} \tag{C.3}$$

$$\phi_t\mathcal{P}_{x,t}=\mathrm{D}_{x,t}+\left(1-\delta_J\right)\mathbb{E}_t\Xi_{t+1}\phi_{t+1}\mathcal{P}_{x,t+1} \tag{C.4}$$

$$F_{x,t}=L_{x,t}^{\gamma}Z_{F,x,t}^{1-\gamma} \tag{C.5}$$

$$\gamma\frac{F_{x,t}}{L_{x,t}}=\mathcal{R}_{L,x,t} \tag{C.6}$$

$$\left(1-\gamma\right)\frac{F_{x,t}}{Z_{F,x,t}}=\mathcal{P}_{x,t} \tag{C.7}$$

$$Z_{x,t}=\left[K_{x,t-1}^{\alpha}(A_{x,t}H_{x,t})^{1-\alpha}\right]^{1-\kappa}M_{x,t}^{\kappa} \tag{C.8}$$

$$\left(1-\kappa\right)\alpha\mathcal{P}_{x,t}\frac{Z_{x,t}}{K_{x,t-1}}=\mathcal{R}_{K,x,t} \tag{C.9}$$

$$\left(1-\kappa\right)\left(1-\alpha\right)\mathcal{P}_{x,t}\frac{Z_{x,t}}{H_{x,t}}=W_{x,t} \tag{C.10}$$

$$\kappa \mathcal{P}_{x,t} \frac{Z_{x,t}}{M_{x,t}} = P_{x,t} \tag{C.11}$$

$$K_{x,t} = (1-\delta_K) K_{x,t-1} + \left[1 - \Phi\left(\frac{I_{x,t}}{I_{x,t-1}}\right)\right] I_{x,t} \tag{C.12}$$

$$1 = \mathbb{E}_t \Xi_{t+1} \frac{\mathcal{R}_{K,x,t+1} + Q_{x,t+1}(1-\delta_K)}{Q_{x,t}} \tag{C.13}$$

$$\begin{aligned} P_{x,t} =& Q_{x,t}\left(1 - \Phi\left(\frac{I_{x,t}}{I_{x,t-1}}\right) - \Phi'\left(\frac{I_{x,t}}{I_{x,t-1}}\right)\frac{I_{x,t}}{I_{x,t-1}}\right) \\ &+ \mathbb{E}_t \Xi_{t+1} Q_{x,t+1} \Phi'\left(\frac{I_{x,t+1}}{I_{x,t}}\right)\left(\frac{I_{x,t+1}}{I_{x,t}}\right)^2 \end{aligned} \tag{C.14}$$

$$Y_{x,t} = C_{x,t} + I_{x,t} + M_{x,t} \tag{C.15}$$

$$Z_{x,t} = Z_{F,x,t} + \phi_t\left[J_{x,t} - (1-\delta_J) J_{x,t-1}\right] + J_{x,t}(1+\lambda)^{-\frac{1+\lambda}{\lambda}} \mathcal{P}_{x,t}^{-\frac{1+\lambda}{\lambda}} \overline{Y}_{x,t} \tag{C.16}$$

$$\int_X E_{x,t} \mathrm{d}x = \int_X F_{x,t} \mathrm{d}x \tag{C.17}$$

$$\begin{aligned} U_{x,t} =& \left(\frac{C_{x,t}}{N_{x,t-1}}\right)^{\theta_C}\left(\frac{E_{x,t}}{N_{x,t-1}}\right)^{\theta_F}\left(\frac{1-L_{x,t}}{N_{x,t-1}}\right)^{\theta_L} \cdot \\ & \left(\frac{1}{1+\nu}\Gamma^{1+\nu} - \frac{1}{1+\nu}\left(\frac{H_{x,t}}{N_{x,t-1}}\right)^{1+\nu}\right)^{\theta_H}\left(\Omega - \frac{1}{2}\left(\log\left(\frac{N_{x,t-1}}{N_{t-1}}\right)\right)^2\right)^{\theta_N} \cdot, \\ & \left(1 - \frac{\mathcal{N}_{x,t}}{N_{x,t-1}}\right)^{\psi_1}\left(\overline{d} - \frac{\mathcal{D}_{x,t}}{\mathcal{N}_{x,t}}\right)^{\psi_2} \exp\left[\psi_3 \int_X \frac{N_{\tilde{x},t-1}}{N_{t-1}} \log\frac{\mathcal{N}_{x,\tilde{x},t}}{N_{x,t-1}} \, \mathrm{d}\tilde{x}\right] \end{aligned} \tag{C.18}$$

$$N_t \equiv \int_X N_{\tilde{x},t} \mathrm{d}\tilde{x}, \tag{C.19}$$

$$\mathcal{N}_{x,t} \equiv \int_X \mathcal{N}_{x,\tilde{x},t} \mathrm{d}\tilde{x}, \tag{C.20}$$

$$\mathcal{D}_{x,t} \equiv \int_X d(x,\tilde{x}) \mathcal{N}_{x,\tilde{x},t} \mathrm{d}\tilde{x} \tag{C.21}$$

$$N_{x,t} = G_{N,t} N_{x,t-1} - \mathcal{N}_{x,t} + \int_X \mathcal{N}_{\tilde{x},x,t} \mathrm{d}\tilde{x} \tag{C.22}$$

$$\theta_C E_{x,t} = \theta_F P_{x,t} C_{x,t} \tag{C.23}$$

$$\theta_L E_{x,t} = \theta_F \begin{pmatrix} \mathcal{R}_{L,x,t} - \Phi_L\left(\frac{L_{x,t}}{L_{x,t-1}}\right) - \Phi_{\mathrm{L}}'\left(\frac{L_{x,t}}{L_{x,t-1}}\right)\frac{L_{x,t}}{L_{x,t-1}} \\ +\mathbb{E}_t\left[\Xi_{t+1}\Phi_{\mathrm{L}}'\left(\frac{L_{x,t+1}}{L_{x,t}}\right)\left(\frac{L_{x,t+1}}{L_{x,t}}\right)\right] \end{pmatrix} (1 - L_{x,t}) \tag{C.24}$$

$$\theta_H\left(\frac{H_{x,t}}{N_{x,t-1}}\right)^{\nu} = \theta_F \frac{N_{x,t-1}}{E_{x,t}} W_{x,t}\left(\frac{1}{1+\nu}\Gamma^{1+\nu} - \frac{1}{1+\nu}\left(\frac{H_{x,t}}{N_{x,t-1}}\right)^{1+\nu}\right) \tag{C.25}$$

$$1 = R_t \mathbb{E}\Xi_{t+1} \tag{C.26}$$

$$\Xi_{t+1} = \beta_t \frac{N_{x,t} E_{x,t} U_{x,t+1}^{1-\varsigma}}{E_{x,t+1} N_{x,t-1} U_{x,t}^{1-\varsigma}} \tag{C.27}$$

$$\mu_{N,x,t} = \beta_t \mathbb{E}_t \left[\left[\begin{matrix} \mu_{N,x,t+1} G_{N,t+1} + U_{x,t+1}^{1-\varsigma} \\ +(1-\varsigma) U_{x,t+1}^{1-\varsigma} \left[\begin{matrix} \theta_H \frac{\left(\frac{H_{x,t+1}}{N_{x,t}}\right)^{1+\nu}}{\frac{1}{1+\nu}\Gamma^{1+\nu} - \frac{1}{1+\nu}\left(\frac{H_{x,t+1}}{N_{x,t}}\right)^{1+\nu}} \\ -\theta_N \frac{\log\left(\frac{N_{x,t}}{N_t}\right)}{\frac{1}{2}\Omega^2 - \frac{1}{2}\left(\log\left(\frac{N_{x,t}}{N_t}\right)\right)^2} \\ +\psi_1 \frac{\mathcal{N}_{x,t+1}}{N_{x,t} - \mathcal{N}_{x,t+1}} - (\theta_C + \theta_F + \theta_L + \psi_3) \end{matrix}\right] \end{matrix}\right]\right] \tag{C.28}$$

$$\mu_{N,x,t} = \mu_{N,\check{x},t} + (1-\varsigma)\, N_{x,t-1} U_{x,t}^{1-\varsigma} \begin{bmatrix} \psi_3 \frac{N_{\check{x},t-1}}{N_{t-1}\mathcal{N}_{x,\check{x},t}} - \psi_1 \frac{1}{N_{x,t-1} - \mathcal{N}_{x,t}} \\ -\psi_2 \frac{d(x,\check{x})\mathcal{N}_{x,t} - \mathcal{D}_{x,t}}{\overline{d}\mathcal{N}_{x,t}^2 - \mathcal{N}_{x,t}\mathcal{D}_{x,t}} \end{bmatrix} \tag{C.29}$$