A Robust Monetary Rule

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Abstract: Central banks would like to ensure determinate inflation, to rule out self-fulfilling fluctuations. Traditional monetary rules can fail to produce determinacy under a variety of conditions. This paper proposes a family of monetary rules that ensure determinate inflation under the weakest possible assumptions about the behaviour of households and firms. Despite this, the family of rules is general enough to allow the determinate implementation of arbitrary inflation dynamics. The rules are easy to implement in practice, and even simple rules in our class attain high welfare. Existing US Fed behaviour is close to one such simple rule.

Keywords: robust monetary rules, determinacy, Taylor principle, inflation dynamics

JEL codes: E52, E43, E31

Today you start work as president of the Fictian Central Bank (FCB). As FCB president, you have a clear mandate to stabilize inflation, even if that results in unemployment or output losses. How should you act? You have studied New Keynesian macro, so you are inclined to follow some variant of the Taylor rule. You recall the prescription of the Taylor principle: the response of nominal rates to inflation should be greater than one to ensure determinacy and rule out self-fulfilling fluctuations in inflation. But you also remember reading other papers which talked of the Taylor principle being insufficient if there are hand-to-mouth households (Gali, Lopez-Salido & Valles 2004), firm-specific capital (Sveen & Weinke 2005), high government spending (Natvik 2009), or if the inflation target is positive (Ascari & Ropele 2009), particularly in the presence of trend growth and sticky wages (Khan, Phaneuf & Victor 2019). Indeed, you recollect that the Taylor principle inverts if there are sufficiently many hand-to-mouth households (Bilbiie 2008), certain financial frictions (Manea 2019), or non-rational expectations (Branch & McGough 2010; 2018). Is there a way you could act to ensure determinacy even if one or more of these circumstances is true? This paper argues that there is.

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To illustrate the idea behind our rules, suppose that both nominal and real bonds are traded in an economy. If a unit of the former is purchased at t, it returns the principal plus a nominal yield of i_t in period $t+1$. If a unit of the latter is purchased at t, it returns the principal plus a nominal yield of $r_t + \pi_{t+1}$ in period $t+1$, where π_{t+1} is realized inflation between t and $t+1$. Abstracting for the moment from inflation risk premia, term premia and liquidity premia, arbitrage between these two markets implies that the Fisher equation must hold, i.e.:

$$i_t = r_t + \mathbb{E}_t \pi_{t+1}, \tag{1}$$

where $\mathbb{E}_t \pi_{t+1}$ is the full information rational expectation of period $t+1$'s inflation rate, given period t's information. Suppose further than the central bank observes both the nominal and real bond markets, and that it can intervene in the former. Then the central bank can choose to set nominal interest rates according to the simple rule:[2]

$$i_t = r_t + \phi \pi_t, \tag{2}$$

where $\phi > 1$. Combining these two equations gives that:

$$\mathbb{E}_t \pi_{t+1} = \phi \pi_t,$$

which has a unique non-explosive solution of $\pi_t = 0$.[3] Determinate inflation!

Why is this robust? Firstly, the rule does not require the aggregate Euler equation to hold, even approximately. For the Fisher equation (1) to hold (still ignoring risk/term/liquidity premia for now), there only need to be two deep pocketed, fully informed, rational agents. Arbitrage takes care of the rest. Even full information is not necessary. Since large markets aggregate information (Hellwig 1980; Lou et al. 2019), the Fisher equation can come to hold even when information about future inflation is dispersed amongst market participants.

[2] Such rules have appeared in Adão, Correia & Teles (2011), Holden (2019) and Lubik, Matthes & Mertens (2019) amongst other places. However, in the prior literature they have chiefly been introduced for analytic convenience, rather than as serious proposals. The one exception is Cochrane (2017) who briefly discusses a rule of this form before moving on to discuss rules which hold $i_t - r_t$ constant (i.e. rules with $\phi = 0$). Cochrane (2018) further explores rules holding $i_t - r_t$ constant. The "indexed payment on reserve" rules of Hall & Reis (2016) also rely on observable real rates, but use a different mechanism to achieve determinacy. They propose that the CB issues an asset ("reserves") with nominal return from \$1 of $\$(1 + r_t)\frac{p_{t+1}}{p_t^*}$ or $\$(1 + i_t)\frac{p_t}{p_t^*}$.

Additionally, in older work, Hetzel (1990) proposes using the spread between nominal and real bonds to guide monetary policy, and Dowd (1994) proposes targeting the price of futures contracts on the price level, which has a similar flavour to our rules, since our rules effectively use expected inflation as the instrument of monetary policy. There is also an established literature looking at rules tracking the efficient real interest rate, see e.g. Cúrdia et al. (2015), which is a very different idea.

[3] Here we sidestep the issues raised by Cochrane (2011) and follow the standard New Keynesian literature in assuming agents will always select non-explosive paths for inflation. The escape clause rules of Christiano & Takahashi (2018) are one way by which central banks could ensure coordination on the expectations consistent with non-explosive inflation.

Given that the rule does not require the aggregate Euler equation to hold, it is automatically robust to heterogeneity, hand-to-mouth agents and non-rational consumer expectations. The only expectations that matter are the expectations of participants in the markets for nominal and real bonds. It is surely much more reasonable to assume that financial market outcomes lead to rational expectations than to assume rationality of households more generally.

Secondly, the rule does not require the aggregate Phillips equation to hold. The slope of the Phillips curve will have no impact on the dynamics of inflation. If the FCB president is unconcerned with output, they do not need to know if the Phillips curve holds, let alone its slope. Nor does it matter how firms form inflation expectations. Inflation is pinned down by the Fisher and monetary rules, so while non-rational firm expectations could affect output fluctuations, they will not alter the dynamics of inflation.

This may be surprising. How could price setters fail to determine inflation? The short answer is "Walras's law". To see how this plays out, suppose that today all firms decide to double their price. Financial market participants still expect zero inflation next period, because that is the only outcome consistent with non-explosive inflation in future. Thus, financial market participants always value nominal bonds the same as real bonds. But the central bank's monetary rule instructs it to attempt to produce nominal rates which are much higher than real rates, as today's inflation is high. So, the central bank wants to sell nominal bonds, i.e. to borrow money from financial market participants.

However, no amount of nominal bond selling will induce market participants to lower their valuation of nominal bonds below that of real bonds, though both valuations may fall together (i.e. both nominal and real rates rise). Thus, the central bank will end up reducing the money supply to zero. With households having zero cash, not all final goods will be sold.[4] Thus, the final goods market does not clear. To obtain market clearing in final goods, at least some price setters must reduce their price until inflation is zero, so ensuring that the central bank sets nominal rates equal to real rates.

The rest of this paper further examines rules responding to real rates. The next section generalizes the simple rule of equation (2), and goes on to show that there are similar rules that determinately implement an arbitrary path for inflation, robustly across models. Section 2 discusses how rules of these forms could be implemented in practice. Next, Section 3 looks at the welfare costs of these rules when their generality is limited, in various models. Section 4 looks at the data to see how close existing

[4] For example, if there are cash goods and credit goods, only credit goods will be sold.

central bank behaviour is to following a rule of our proposed form. Finally, Section 5 looks at the consequences of the zero lower bound for the performance of these rules.

1 Generalizations and generality

While the simple rule (2) always produces zero inflation, slight extensions of the rule allow inflation to move. For example, we may add a monetary policy shock, ζ_t to the rule, giving:

$$i_t = r_t + \phi\pi_t + \zeta_t. \tag{3}$$

Monetary policy shocks may perhaps reflect the central bank's limited information. If the central bank does not perfectly observe current inflation, and sets interest rates to $i_t = r_t + \phi\tilde{\pi}_t$, where $\tilde{\pi}_t$ is its signal about inflation, then it will end up setting a slightly different level for $i_t - r_t$ to that dictated by the rule with $\nu_t = 0$, effectively generating monetary policy shocks.[5]

The central bank might also deliberately decide to introduce monetary policy shocks correlated with the economy's structural shocks. For example, by lowering $i_t - r_t$ following a positive mark-up or cost-push shock, the central bank can lessen the movement in the output gap.[6] This has no effect on the determinacy region as structural shocks are exogenous. For now though, we assume that ζ_t is independent of other structural shocks.

From combining (3) with the Fisher equation (1) we have:

$$\mathbb{E}_t\pi_{t+1} = \phi\pi_t + \zeta_t,$$

which (with $\phi > 1$) has the unique solution $\pi_t = -\frac{1}{\phi-\rho}\zeta_t$, if ζ_t follows an AR(1) process with persistence ρ.

A contractionary (positive) monetary policy shock results in a fall in inflation, as expected. If the central bank is more aggressive, so ϕ is larger, then inflation is less volatile. Only monetary policy shocks affect inflation. Of course, if there is a nominal rigidity in the model, such as sticky prices or wages, monetary shocks may have an impact on real variables. But as long as the central bank follows rules like this, these real disruptions have no feedback to inflation. We can understand inflation without worrying about the rest of the economy.

In line with this, an extensive body of empirical evidence finds no role for the Phillips curve in forecasting inflation (see e.g. Atkeson & Ohanian 2001; Ang, Bekaert & Wei

[5] Lubik, Matthes & Mertens (2019) look at the determinacy consequences of a central bank that filters inflation signals in order to retrieve the optimal estimate. The determinacy problems they highlight all disappear if the central bank directly responds to its signal.

[6] Ireland (2007) presents evidence that the US Federal Reserve has reacted to mark-up shocks.

2007; Stock & Watson 2009; Dotsey, Fujita & Stark 2018). In a recent contribution, Dotsey, Fujita & Stark (2018) find that in the post-1984 period, Phillips curve based forecasts perform worse than those of a simple IMA(1,1) model, both unconditionally and conditional on various measures of the state of the economy. This provides strong support for models in which the causation in the Phillips curve runs in only one direction: from inflation to the output-gap.[7]

To see how this emerges under the monetary rule (3), suppose the rest of the model comprises the Phillips curve:[8]

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \kappa \omega_t, \tag{4}$$

and the discounted/compounded Euler equation:

$$x_t = \delta \mathbb{E}_t x_{t+1} - \varsigma (r_t - n_t), \tag{5}$$

where x_t is the output gap, ω_t is a mark-up/cost-push shock, and n_t is the exogenous natural real rate of interest. This form of discounted/compounded Euler equation appears in Bilbiie (2019) and (under discounting) in McKay, Nakamura & Steinsson (2017). The latter paper shows it provides a good approximation to a heterogeneous agent model with incomplete markets. The standard Euler equation is recovered if $\delta = 1$ and ς is the elasticity of intertemporal substitution. This specification also nests the limited asset market participation or "TANK" model of Bilbiie (2008) when $\delta = 1$, but ς is allowed to be negative.

Since $\pi_t = -\frac{1}{\phi - \rho}\zeta_t$, with ζ_t being AR(1) with persistence ρ, the Phillips curve (4) implies that $x_t = -\frac{1}{\kappa}\frac{1-\beta\rho}{\phi-\rho}\zeta_t - \omega_t$. Does x_t help forecast π_t here? Clearly no. $\mathbb{E}_t \pi_{t+1} = -\frac{1}{\phi-\rho}\mathbb{E}_t \zeta_{t+1} = -\frac{\rho}{\phi-\rho}\zeta_t = \rho \pi_t$. Once you know π_t, you already have all the information you need to form the optimal forecast of π_{t+1}. The correlation in π_t and x_t provides no extra information.[9]

This model also enables us to show the robustness of our rule's determinacy in practice. Note that with x_t expressed as a linear combination of exogenous variables, there is no need to solve the Euler equation (5) forward, so the degree of discounting (δ) can have no effect on determinacy. Not needing to solve the Euler equation forward also gives robustness to a missing transversality constraint on household assets. For example, if ω_t is independent across time, then the Euler equation implies $r_t = n_t +$

[7] McLeay & Tenreyro (2019) provide an alternative explanation based on the fact that optimal policy prescribes a negative correlation between inflation and output, making difficult empirical identification of the Phillips curve.
[8] Throughout this paper, we multiply the mark-up shock by κ as the ratio of the response to x_t and the response to ω_t is not a function of either the (Calvo) price adjustment probability or the (Rotemberg) price adjustment cost. See Khan (2005) for derivations.
[9] This result is robust to generalizing to an ARMA(1,1) process for ζ_t. See Appendix A.1.

$\frac{1}{\varsigma}\left[\frac{1}{\kappa}\frac{(1-\beta\rho)(1-\delta\rho)}{\phi-\rho}\zeta_t+\omega_t\right]$. This contrasts with the results of Bilbiie (2019) who found that the Taylor principle ($\phi>1$) was only sufficient for determinacy in the discounting case ($\delta\le 1$), and with Bilbiie (2008) who found that the Taylor principle ($\phi>1$) was only sufficient in the $\varsigma>0$ case. Under our rule (3), the Taylor principle is necessary and sufficient for determinacy whether there is discounting or compounding, and whether ς is positive or negative (given $\phi\ge 0$).[10]

In fact, variants of our rule can determinately implement any path for inflation, no matter the rest of the model. Let π_t^* be an exogenous stochastic process, perhaps a function of the economy's other shocks,[11] and consider the rule:

$$i_t = r_t + \mathbb{E}_t\pi_{t+1}^* + \phi(\pi_t - \pi_t^*). \tag{6}$$

From the Fisher equation (1), this implies:

$$\mathbb{E}_t(\pi_{t+1} - \pi_{t+1}^*) = \phi(\pi_t - \pi_t^*).$$

Again with $\phi>1$, there is a unique solution, now with $\pi_t=\pi_t^*$. I.e., at all periods of time, and in all states of the world, realised inflation is equal to π_t^*. Effectively, the central bank is able to choose an arbitrary path for inflation as the unique, determinate equilibrium outcome.

The only constraint is that the targeted path for inflation cannot be a function of endogenous variables. However, this is not much of a limitation, since in stationary equilibrium, endogenous variables must have a representation as a function of the infinite history of the economy's shocks. This means that by choosing π_t^* appropriately, rules in the form of (6) can mimic the outcomes of any other monetary policy regime.[12]

For example, suppose that the central bank were to set interest rates in a different (though time invariant) way, for example by using another rule, or by adopting optimal policy under either commitment or discretion, given some objective. For simplicity, suppose further that the economy's equilibrium conditions are linear, e.g. because we are working under a first order approximation. Let $(\varepsilon_{1,t},\dots,\varepsilon_{N,t})_{t\in\mathbb{Z}}$ be the set of structural shocks in the economy,[13] all of which are assumed mean zero and independent both of each other, and over time. Finally, assume that the central bank's

[10] In Appendix A.2 we prove that this is robust to monetary responses to the real rate which are not exactly equal to 1.

[11] Ireland (2007) also allows the central bank's inflation target to respond to other structural shocks.

[12] Other papers have examined the implementation of optimal policy in specific models using instrument rate rules (see e.g. Svensson & Woodford 2003; Dotsey & Hornstein 2006; Evans & Honkapohja 2006; Evans & McGough 2010). However, the various prior proposals do not enable the implementation of a certain inflation path robustly across models.

[13] This may include sunspot shocks if they are added following Farmer, Khramov & Nicolò (2015).

behaviour produces stationary inflation, $\tilde{\pi}_t$, with the $\widetilde{\ }$ denoting that this is inflation under the alternative monetary regime. Then, by linearity and stationarity, there must exist a constant $\tilde{\pi}^*$ and coefficients $(\theta_{1,k}, \dots, \theta_{N,k})_{k \in \mathbb{N}}$ such that:

$$\tilde{\pi}_t = \tilde{\pi}^* + \sum_{k=0}^{\infty} \sum_{n=1}^{N} \theta_{n,k} \varepsilon_{n,t-k},$$

with $\sum_{k=0}^{\infty} \theta_{n,k}^2 < \infty$ for $n = 1, \dots, N$. So, if the central bank sets:

$$\pi_t^* = \tilde{\pi}^* + \sum_{k=0}^{\infty} \sum_{n=1}^{N} \theta_{n,k} \varepsilon_{n,t-k},$$

(exogenous!) and uses the rule (6), then for all t and in all states of the world, $\pi_t = \pi_t^* = \tilde{\pi}_t$. Moreover, this implies in turn that all the endogenous variables in the two economies must be identical in all periods and in all states of the world.[14]

The only slight difficulty with this approach is that the central bank may struggle to observe structural shocks. The central bank can certainly observe linear combinations of structural shocks, via estimating a VAR with sufficiently many lags. For variables that are plausibly contemporaneously exogenous, such as commodity prices for a small(ish) economy, this is already sufficient to recover the corresponding structural shock. To infer other shocks, the central bank needs to know more about the structure of the economy. However, we do not need to assume any more than is standard in rational expectations models. Forming rational expectations requires you to know the structure of the economy; if you know this structure, then you know the mapping from the reduced form shocks estimated by a VAR to the model's structural shocks.[15]

2 Practical implementation

Until recently, central banks concentrated their monetary interventions in overnight debt markets. However, with the rise of quantitative easing, many central banks have been purchasing substantial quantities of longer maturity sovereign debt. There is no reason then that central banks could not conduct open market operations to fix the interest rate on longer maturity bonds. This is convenient as in most countries, inflation protected securities are only issued at long maturities, e.g. five years.

In practice then, the central bank's trading desk would be tasked with maintaining a particular level of the gap between nominal and real rates according to the market for bonds of a certain maturity. For the rest of this section, we shall assume five-year bonds

[14] Proven in Appendix A.3.

[15] This mapping may not be unique valued if there are more shocks than observables. However, since we expect a relatively small number of shocks to explain the bulk of business cycle variance, this is unlikely to be problematic in practice.

are used, since five-year treasury inflation protected securities (TIPS) are the shortest maturity available in the US.

So, let i_t be the nominal yield per-period on a five-year sovereign bond at t, and r_t be the real yield per-period on a five-year inflation protected bond from the same issuer. As ever, t indexes time. The units of time do not need to coincide with the maturity of the bond. In particular, t may be measured in months, quarters or years, in which case i_t is the nominal yield per-month, per-quarter or per-year, respectively. Let T be the number of periods in five years. For example, T may be 60 if periods are months.

We also allow for the possibility that inflation is not observed contemporaneously. For example, US CPI is observed with a one-month lag. To capture this, while keeping to the convention that $\mathbb{E}_t v_t = v_t$ for all t-dated endogenous variables v_t, we assume that market participants and the central bank use the $t - L$ information set in period t (i.e. they know the values of all $t - L, t - L - 1, \ldots$ dated variables), for some $L \geq 0$. Thus, since the central bank does not know π_t at t, we instead assume that they respond to deviations of π_{t-L} from target, rather than π_t.

We allow for a shock in the Fisher equation to capture inflation risk premia, liquidity premia, asymmetric term premia and even further departures from full information rational expectations amongst market participants. Since only $t - L$ dated variables are known in period t, we denote the period t value of this shock by ν_{t-L}. I.e. risk premia (etc.) will be determined L periods in advance, though market participants and the central bank will not act on this, since they use L period old data. Given this, the Fisher equation coming from arbitrage between nominal and real bonds then states that:

$$i_t - r_t = \nu_{t-L} + \mathbb{E}_{t-L} \frac{1}{T} \sum_{k=1}^{T} \pi_{t+k},$$

where ν_{t-L} is the aforementioned shock to risk premia (etc.). We only require that ν_t is a stationary process.

Slightly generalizing our previous rule (6), we suppose that the central bank intervenes in five-year nominal bond markets to ensure that it is always the case that:

$$i_t - r_t = \bar{\nu}_{t-L} + \mathbb{E}_{t-L} \frac{1}{T} \sum_{k=1}^{T} \pi_{t+k}^* + \phi(\pi_{t-L} - \pi_{t-L}^*),$$

where $\bar{\nu}_{t-L}$ is the central bank's period t belief about the level of ν_{t-L}.

We have deliberately not added any interest rate smoothing. While such smoothing is often believed to be a relevant feature of real-world central bank behaviour, in our context it adds nothing. Smooth paths for interest rates may be produced from a smooth target path for π_t^*.[16]

Also note that while under conventional monetary policy, targeted nominal interest rates are (approximately) constant between monetary policy committee meetings, this may not be the case here. The rule effectively specifies a period t level for $i_t - r_t$, not for i_t. The level of r_t may fluctuate (perhaps in part due to unexpected changes in i_t), so the central bank's trading desk could have to continuously tweak the level of i_t to hold $i_t - r_t$ at its desired level. While this represents a departure from previous operating procedure, there is no reason why holding $i_t - r_t$ approximately constant should be any harder than holding i_t approximately constant. This is thanks to real-time observability of r_t via inflation protected bonds. The central bank could also directly control $i_t - r_t$ by promising to freely exchange \$1 face value of real debt for $\$(1 + i_t - r_t)$ face value of nominal debt, as suggested by Cochrane (2017; 2018). Alternatively, the central bank could buy or sell a long-short portfolio containing \$1 face value of nominal debt, and $-$\$1 face value of real debt to hold the portfolio's price fixed at $\$(i_t - r_t)$.[17]

Thus, the monetary rule implies that the dynamics of inflation are governed by the single equation:

$$\mathbb{E}_{t-L}\frac{1}{T}\sum_{k=1}^{T}(\pi_{t+k} - \pi_{t+k}^*) = (\bar{\nu}_{t-L} - \nu_{t-L}) + \phi(\pi_{t-L} - \pi_{t-L}^*),$$

i.e.:

$$\mathbb{E}_t\frac{1}{T}\sum_{k=1}^{T}(\pi_{t+k+L} - \pi_{t+k+L}^*) = (\bar{\nu}_t - \nu_t) + \phi(\pi_t - \pi_t^*).$$

As ever, with $\phi > 1$, there is a unique solution.[18] In the special case in which the central bank observes ν_t (i.e. risk premia etc.) so $\nu_t = \bar{\nu}_t$, then $\pi_t = \pi_t^*$, as before. In the general case, as long as $\bar{\nu}_t - \nu_t$ is stationary, the solution takes the form:[19]

$$\pi_t = \pi_t^* + \mathbb{E}_t\sum_{j=0}^{\infty}A_j\big(\bar{\nu}_{t+j} - \nu_{t+j}\big),$$

[16] In situations in which the dynamics of π_t^* are constrained, then adding smoothing may help match real-world dynamics. In this case, the independence of inflation from the rest of the economy can be preserved if rather than i_{t-1} appearing on the right hand side of the monetary rule, instead there is $i_{t-1} - r_{t-1}$.

[17] The author thanks Peter Ireland for this suggestion.

[18] We do not have the indeterminacy issues for rules setting long-rates that were noted by McGough, Rudebusch & Williams (2005), due to the presence of the real rate in our rule.

[19] Ireland (2015) finds a role for risk premia in explaining US inflation fluctuations, so risk premia appearing in the solution for inflation should not be too surprising.

where $A_0 := -\frac{1}{\phi}$, $A_j := 0$ for $j \in \{1, \dots, L\}$, and $A_j := \frac{1}{\phi T}\sum_{k=\max\{0,j-L-T\}}^{j-L-1} A_k$ for all $j > L$, implying $A_{L+1} = -\frac{1}{T\phi^2}$ and $A_j = \mathrm{O}\left(\phi^{-\frac{j}{T+L}}\right)$ as $j \to \infty$.[20] Thus, with ϕ large, even if the central bank imperfectly tracks the risk (etc.) premium ν_t, it will still be the case that $\pi_t \approx \pi_t^*$ in all periods. I.e., even in the presence of unobservable risk premia, the central bank can still determinately implement an arbitrary path for inflation. The presence of information lags makes no fundamental difference to this. While information lags may slow down the convergence of A_j to 0 as $j \to \infty$, increasing the variance of $\pi_t - \pi_t^*$, still for a large enough ϕ, inflation will be very close to its target.

3 Welfare

In Section 1, we established that a rule of our form could exactly mimic any other time invariant policy, if responses to structural shocks and their lags are allowed. Thus, rules of our form can mimic e.g. unconditionally optimal policy, optimal commitment policy from a timeless perspective, or optimal discretionary policy. Hence, rules of our form are capable of achieving high levels of welfare.

We begin this section by looking at unconditionally optimal time-invariant policy using our rules, in a simple model. We then go on to analyse the performance of our rules if further restrictions are placed upon them, such as only permitting the central bank to respond to current or sufficiently recent shocks. We show that optimal policy in estimated models of the US economy comes close to stabilizing inflation, with optimal inflation dynamics describable by an ARMA process with few MA terms.

Any welfare analysis requires us to specify the rest of the model, as welfare is generally a function of output's variability, not just that of inflation. Thus, as a first example suppose that inflation and output are linked by the standard Phillips curve:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \kappa \omega_t,$$

where x_t is the output gap, and ω_t is a mark-up shock, which is assumed IID with mean zero. Additionally, suppose that the policy maker wants to minimise the unconditional expectation of a quadratic loss function in inflation and the output gap. I.e. the period t policy maker minimises:

$$(1-\beta)\mathbb{E}\sum_{k=0}^{\infty}\beta^k\left(\pi_{t+k}^2 + \lambda x_{t+k}^2\right),$$

for some $\lambda > 0$ and $\beta \in (0,1)$.

[20] Guess $A_j \propto B^j$. Then (for large j): $B^j = \frac{1}{\phi T}\sum_{k=j-L-T}^{j-L-1} B^k = \frac{1}{\phi T}\frac{B^{j-L-T}-B^{j-L}}{1-B}$, so $\phi T B^{T+L} = \frac{1-B^T}{1-B} \in [1, T]$, implying $0 \le B \le \phi^{-\frac{1}{T+L}}$.

We suppose that the policy maker is constrained to choose a time-invariant (i.e. stationary) policy, thus the objective simplifies to:[21]

$$\mathbb{E}\big(\pi_t^2 + \lambda x_t^2\big).$$

As the policy maker only cares about inflation and output gaps, with the former being effectively under their control, and the latter only determined by inflation and mark-up shocks, the optimal policy must have the form:

$$\pi_t = \kappa \sum_{k=0}^{\infty} \theta_k \omega_{t-k},$$

for some $\theta_0, \theta_1, \dots$ to be determined. We have already shown that such a policy may be determinately implemented via a rule of the form of (6).

Substituting this policy into the Phillips curve then gives:

$$\sum_{k=0}^{\infty} \theta_k \omega_{t-k} = \beta \sum_{k=0}^{\infty} \theta_{k+1} \omega_{t-k} + x_t + \omega_t,$$

so:

$$x_t = \sum_{k=0}^{\infty} (\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])\omega_{t-k}.$$

Hence, the policy maker's objective is to choose $\theta_0, \theta_1, \dots$ to minimise:

$$\mathbb{E}\big(\pi_t^2 + \lambda x_t^2\big) = \mathbb{E}[\omega_t^2] \sum_{k=0}^{\infty} \big[\kappa^2\theta_k^2 + \lambda(\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])^2\big].$$

The first order conditions then give:[22]

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\theta_1 + \frac{\lambda}{\kappa^2}(\theta_1 - \beta\theta_2) - \beta\frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\forall k > 1, \qquad \theta_k + \frac{\lambda}{\kappa^2}(\theta_k - \beta\theta_{k+1}) - \beta\frac{\lambda}{\kappa^2}(\theta_{k-1} - \beta\theta_k) = 0.$$

Unsurprisingly, this agrees with the unconditionally optimal solution given in the prior literature (e.g. Damjanovic, Damjanovic & Nolan (2008)), which satisfies:

$$\pi_t + \frac{\lambda}{\kappa}(x_t - \beta x_{t-1}) = 0,$$

[21] See e.g. Damjanovic, Damjanovic & Nolan (2008).

[22] See Appendix A.4 for the solution of these conditions.

i.e.:

$$\kappa \sum_{k=0}^{\infty} \theta_k \omega_{t-k} + \frac{\lambda}{\kappa}\left[\sum_{k=0}^{\infty} (\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])\omega_{t-k} \right. \\ \left. - \beta \sum_{k=1}^{\infty} (\theta_{k-1} - \beta\theta_k - \mathbb{1}[k-1=0])\omega_{t-k}\right] = 0.$$

To see the equivalence, note that from matching coefficients, this equation holds if and only if the above first order conditions hold. We will present a convenient representation of the solution to these equations below.

Additionally, note that as $\frac{\lambda}{\kappa^2} \to 0$, $\theta_k \to 0$ for all $k \in \mathbb{N}$. In other words, if the central bank does not care about the output gap, then they optimally choose to have constant inflation, i.e. to follow the rule from equation (2). The central bank also chooses constant inflation if the Phillips curve is vertical (i.e. $\kappa = \pm\infty$). In this case, neither inflation nor mark-up shocks have any impact on the output gap.

The first order conditions derived above also enable us to easily solve for optimal unconditional policy under limited memory. For example, if the central bank does not "remember" $\omega_{t-1}, \omega_{t-2}, \ldots$, so uses a rule that is only a function of ω_t at t, then the optimal θ_0 will satisfy the above first order conditions with $\theta_1 = \theta_2 = \cdots = 0$. This means:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - 1) = 0,$$

so $\theta_0 = \frac{\lambda}{\lambda+\kappa^2}$. It turns out that this exactly coincides with the solution under discretion.[23]

If the central bank can "remember" ω_{t-1}, so π_t is an MA(1), then the optimal solution will have:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\theta_1 + \frac{\lambda}{\kappa^2}\theta_1 - \beta\frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0.$$

The solution has $\theta_0 \geq 0$ and $\theta_1 \leq 0$. Thus, the shock increases π_t while reducing $\mathbb{E}_t\pi_{t+1}$, thus dampening the required movement in x_t, from the Phillips curve. We will see that this is already enough to come close to the fully optimal policy.

[23] See Appendix A.5.

Going one step further, if the central bank can also "remember" π_{t-1}, then they can choose interest rates to ensure π_t follows the ARMA(1,1) process:

$$\pi_t = \rho\pi_{t-1} + \kappa\theta_0\omega_t + \kappa\theta_1\omega_{t-1},$$

for some ρ, θ_0, θ_1 to be determined.[24] Since US inflation appears to be well approximated by an ARMA(1,1) (Stock & Watson 2009), this may be a reasonable model of Fed behaviour. This ARMA(1,1) process has the MA(∞) representation:

$$\pi_t = \kappa\theta_0 \sum_{k=0}^{\infty} \rho^k \omega_{t-k} + \kappa\theta_1 \sum_{k=0}^{\infty} \rho^k \omega_{t-1-k} = \kappa\theta_0\omega_t + \kappa(\rho\theta_0 + \theta_1) \sum_{k=1}^{\infty} \rho^{k-1}\omega_{t-k}. \tag{7}$$

Substituting this policy into the Phillips curve gives:

$$\theta_0\omega_t + (\rho\theta_0 + \theta_1)\sum_{k=1}^{\infty}\rho^{k-1}\omega_{t-k} = \beta(\rho\theta_0 + \theta_1)\omega_t + \beta(\rho\theta_0 + \theta_1)\sum_{k=1}^{\infty}\rho^k\omega_{t-k} + x_t + \omega_t,$$

so:

$$x_t = [(1-\beta\rho)\theta_0 - \beta\theta_1 - 1]\omega_t + (1-\beta\rho)(\rho\theta_0 + \theta_1)\sum_{k=1}^{\infty}\rho^{k-1}\omega_{t-k}.$$

Hence, the policy maker's objective is to choose ρ, θ_0, θ_1 to minimise:

$$\begin{aligned}\mathbb{E}\left(\pi_t^2 + \lambda x_t^2\right) = \mathbb{E}[\omega_t^2]\Bigg[&\kappa^2\theta_0^2 + \lambda[(1-\beta\rho)\theta_0 - \beta\theta_1 - 1]^2 \\ &+ [\kappa^2(\rho\theta_0+\theta_1)^2 + \lambda(1-\beta\rho)^2(\rho\theta_0+\theta_1)^2]\frac{1}{1-\rho^2}\Bigg].\end{aligned}$$

Tedious algebra gives that the first order conditions have solution:[25]

$$\rho = \frac{\kappa^2 + (1+\beta^2)\lambda - \sqrt{(\kappa^2 + (1-\beta)^2\lambda)(\kappa^2 + (1+\beta)^2\lambda)}}{2\beta\lambda}, \qquad \theta_0 = \frac{\rho}{\beta}, \qquad \theta_1 = -\rho.$$

As $\lambda \to 0$, or $\kappa \to \infty$, $\rho \to 0$. As $\lambda \to \infty$, or $\kappa \to 0$, $\rho \to \beta$. Since there is no other solution for κ to the equation $\rho = \beta$ than $\kappa = 0$, we must have $\rho \leq \beta$, so $\rho\theta_0 + \theta_1 \leq 0$, meaning that the response of inflation to a positive mark-up shock is again negative after the

[24] The targeted inflation can respond to lagged targeted inflation without changing the determinacy properties of realised inflation (always equal to targeted inflation in equilibrium). Targeted inflation cannot respond to other endogenous variables without changing these determinacy properties.

[25] There is an additional solution to the first order condition with $\rho = \frac{\kappa^2+(1+\beta^2)\lambda+\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda}$, but this is outside of the unit circle as: $\frac{\kappa^2+(1+\beta^2)\lambda+\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda} > \frac{\kappa^2+(1+\beta^2)\lambda+\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1-\beta)^2\lambda)}}{2\beta\lambda} = \frac{\kappa^2+(1-\beta+\beta^2)\lambda}{\beta\lambda} > \frac{1-\beta+\beta^2}{\beta} = \frac{1}{\beta} + \beta - 1 > 1$. However, the given solution is inside the unit circle as $\frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda} > \frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1+\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda} = -1$, $\frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda} < \frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1-\beta)^2\lambda)}}{2\beta\lambda} = 1$.

first period. Since we have one extra degree of freedom, this must attain even higher welfare than the MA(1) solution. In fact, it attains the unconditionally optimal solution. Examination of the unconditionally optimal solution from Appendix A.4 reveals that it has the same form as equation (7), thus by a revealed preference argument, the two solutions must coincide. (For example, the solution for ρ agrees with the geometric decay rate of the MA coefficients at lags beyond the first of the fully optimal solution we found in Appendix A.4.)

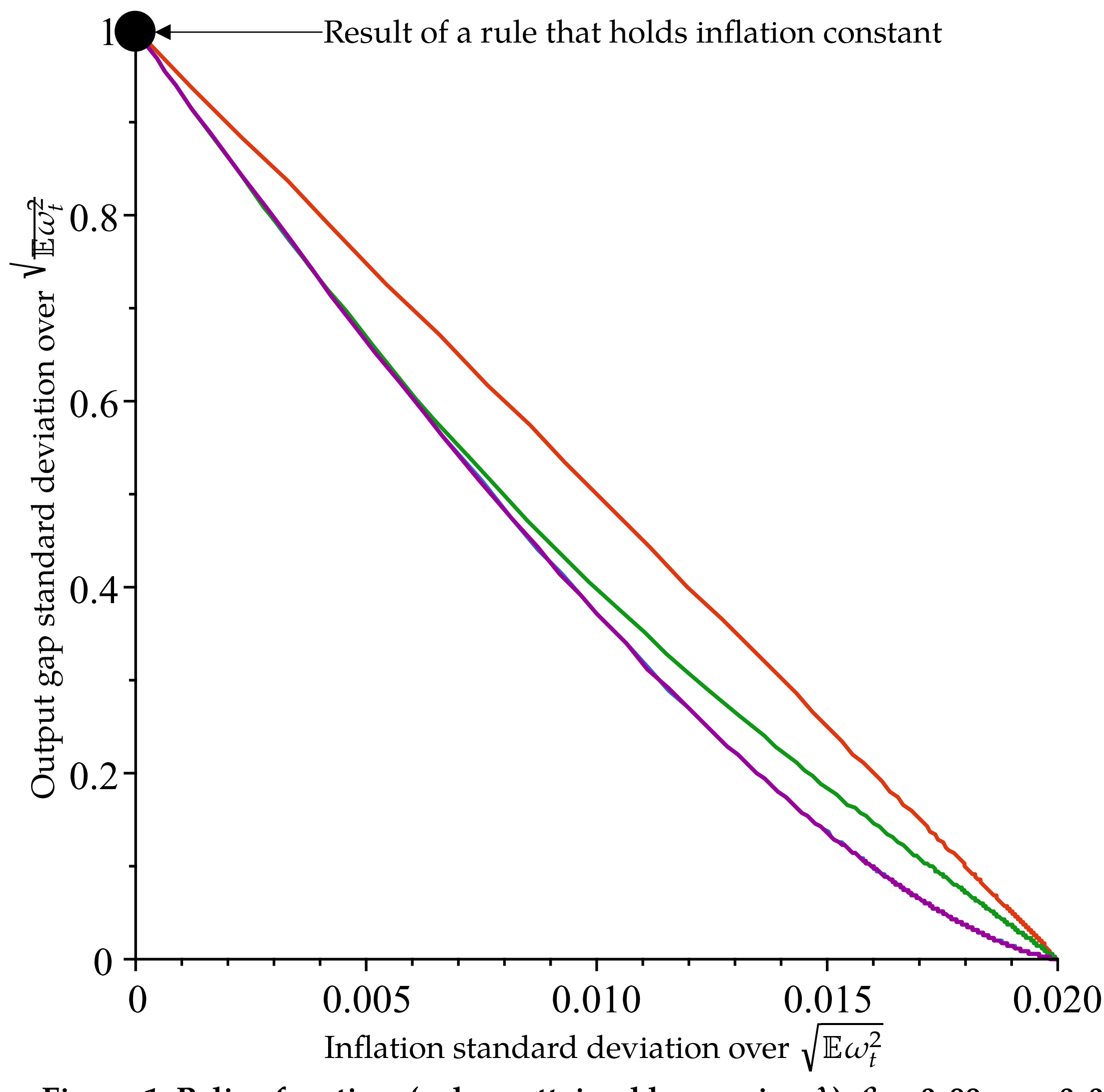

Figure 1: Policy frontiers (values attained by varying λ). $\beta = 0.99$, $\kappa = 0.02$.
Purple: Unconditionally optimal policy, equivalent to ARMA(1, 1) policy.
Blue (hidden behind purple): Timeless optimal solution.
Red: Policy just responding to current shocks, equivalent to discretion.
Green: Policy that responds to current and once lagged shocks.

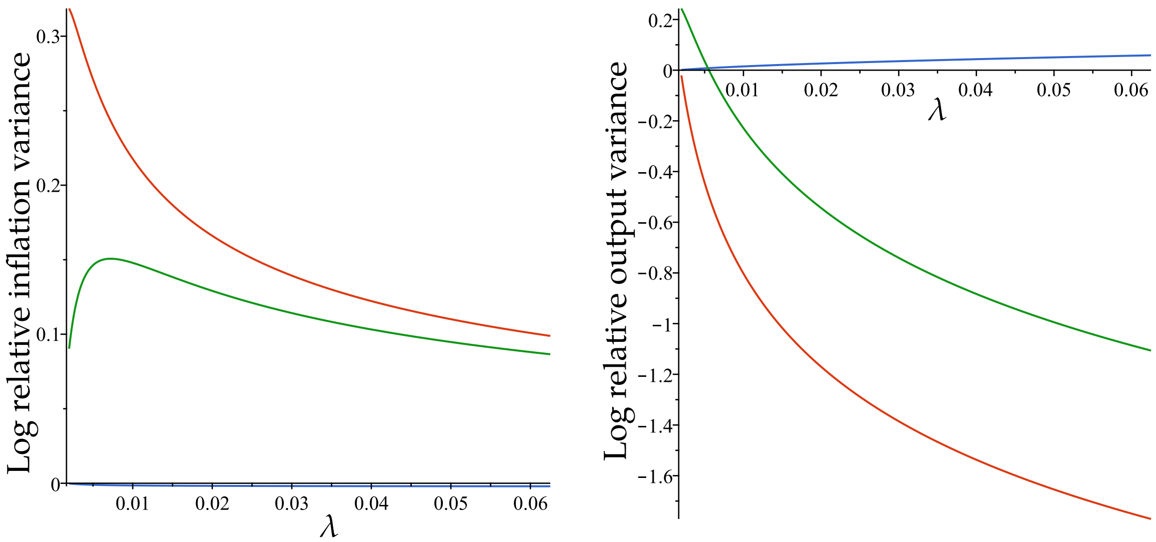

Figure 2: Logarithms of ratios of variance under a given policy to variance under unconditionally optimal policy. $\beta = 0.99$, $\kappa = 0.02$.
Blue: Timeless optimal solution.
Red: Policy just responding to current shocks, equivalent to discretion.
Green: Policy that responds to current and once lagged shocks.

Hence, in a world in which the only inefficient shocks are IID cost-push shocks, the central bank can attain the unconditionally optimal welfare by ensuring inflation follows an appropriate ARMA(1,1) process. This process will have an MA coefficient equal to $-\beta \approx -0.99$, and as long as the central bank cares about output stabilisation, it will have a high degree of persistence. This is very close to the IMA(1,1) processes estimated by Dotsey, Fujita & Stark (2018) for the post-1984 period.

To see the welfare attained by the other policies we have discussed, Figure 1 plots the policy frontiers attained by varying λ for each of the polices. In all cases, we follow Eggertsson & Woodford (2003) in setting $\beta = 0.99$ and $\kappa = 0.02$. The figure makes clear that the MA(1) policy (green) is a substantial improvement on the MA(0) (discretionary) policy (red). It also shows just how close Woodford's timeless perspective (1999)[26] (blue, hidden behind purple) comes to the unconditionally optimal policy. Figure 2 shows how these differences across policies are driven by λ, by plotting the logarithm of the ratio of variance under a given policy to the variance under unconditionally optimal policy. We allow λ to vary from 0.002 (the value obtained by a second order approximation to the consumer's utility with $\kappa = 0.02$, if the elasticity of substitution across goods equals 10) to $\frac{1}{16}$ (corresponding to an equal weight on annual inflation and the output gap). Both the MA(0) and the MA(1) policy generate too much inflation variance and too little variance in output, relative to the unconditionally optimal solution. However, if the central bank can feasibly respond to

[26] See Appendix A.6 for the derivation of this solution.

ω_t and ω_{t-1} they can probably also respond to π_{t-1}, which is enough to deliver the unconditional optimum.

Even in larger models, optimal inflation dynamics appear to be well approximated by an ARMA process with relatively few MA terms. Figure 3 shows the dynamics of observed and optimal inflation in the Justiniano, Primiceri & Tambalotti (2013) model. (This is a medium-scale New Keynesian DSGE model broadly similar to the model of Smets & Wouters (2007).) While actual inflation is highly persistent, with the same shocks hitting the economy, optimal inflation is far less persistent, with the sample autocorrelation essentially insignificant at 95% after four lags.

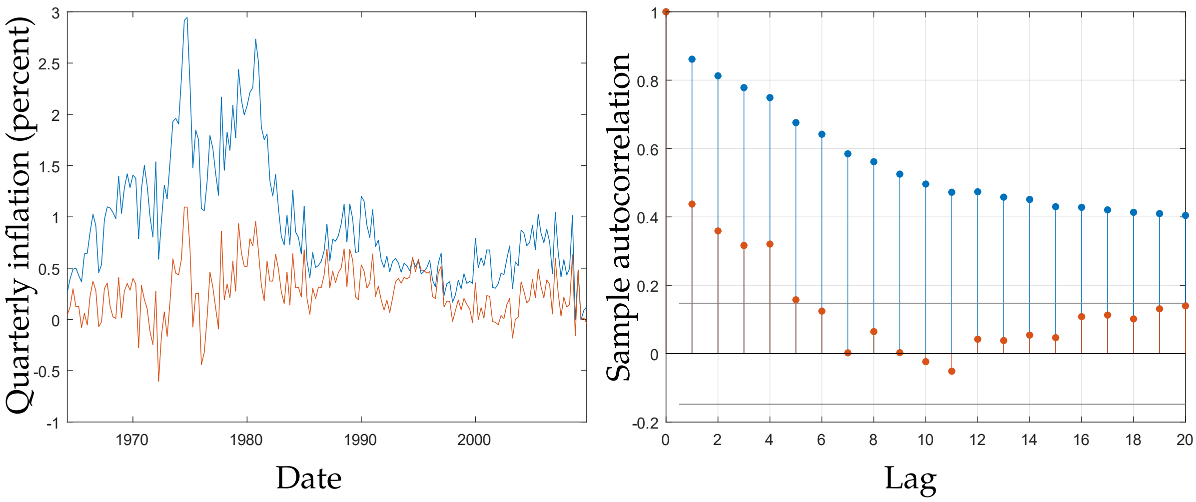

Figure 3: Behaviour of realised inflation (blue) and optimal inflation (red) in the Justiniano, Primiceri & Tambalotti (2013) model.
Left panel shows the timeseries. Right panel shows their sample autocorrelation.

Note that for any $\rho \in (-1,1)$, the solution for optimal inflation has a multiple shock, $\mathrm{ARMA}(1,\infty)$ representation of the form:

$$\pi_t - \pi = \rho(\pi_{t-1} - \pi) + \sum_{k=0}^{\infty} \sum_{n=1}^{N} \theta_{n,k}^{(\rho)} \varepsilon_{n,t-k},$$

where $\varepsilon_{1,t}, \dots, \varepsilon_{N,t}$ are the model's structural shocks. We can approximate this process by truncating the MA terms at some point, e.g. by considering the multiple shock $\mathrm{ARMA}(1,K)$ process:

$$\pi_t^{(K)} - \pi = \rho\big(\pi_{t-1}^{(K)} - \pi\big) + \sum_{k=0}^{K} \sum_{n=1}^{N} \theta_{n,k}^{(\rho)} \varepsilon_{n,t-k}.$$

In Figure 4 we plot the proportion of the variance of optimal inflation that is explained by this truncated process for $K = 0, \dots, 16$, and $\rho \in \{0, 0.61\}$.[27] A multiple shock ARMA(1,1) process already explains over 90% of the variance of optimal inflation, while a multiple shock ARMA(1,2) explains over 95%. Thus, optimal inflation in plausible models can be well approximated by relatively simple inflation dynamics.

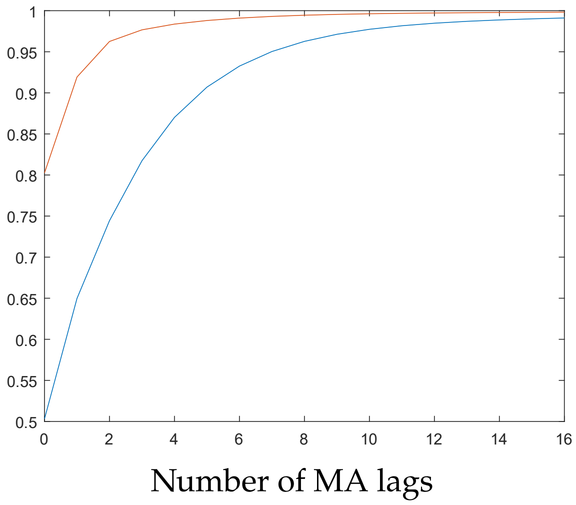

Figure 4: Proportion of the variance of optimal inflation in the Justiniano, Primiceri & Tambalotti (2013) model explained by truncating the number of MA lags. Blue: $\rho = 0$. Red: $\rho = 0.61$.

TODO (MORE, BIGGER MODELS, ETC)

4 Empirical support

In this section we examine whether it is possible that the US Federal Reserve is already using a simple rule of our form. Without the restriction to simple rules, the answer to this would be trivial. Since the rule of equation (6) is compatible with arbitrary inflation dynamics, any observed dynamics can be explained by such a rule.

A natural approach is to take inspiration from the inflation dynamics estimated in reduced form work. Dotsey, Fujita & Stark (2018) find that an IMA(1,1) model for inflation forecasts well, and Stock & Watson (2009) note that this is well approximated by an ARMA(1,1).

Thus, as a first experiment we look at the performance of rules which ensure inflation follows an ARMA(1,1) (as stationarity is convenient), within the benchmark Smets &

[27] $\rho = 0.61$ is the value of ρ that minimises the variance of $\sum_{k=0}^{\infty} \sum_{n=1}^{N} \theta_{n,k}^{(\rho)} \varepsilon_{n,t-k}$. I.e. it is the value of ρ that would be estimated by OLS using an infinite sample of observations from optimal inflation.

Wouters (2007) model of the US economy. In practice then, we just replace the monetary rule from Smets & Wouters (2007) with the equation:

$$\pi_t = \rho^{\mathrm{M}}\pi_{t-1} + \varepsilon_t^{\mathrm{M}} + \theta^{\mathrm{M}}\varepsilon_{t-1}^{\mathrm{M}},$$

where $\varepsilon_t^{\mathrm{M}}$ is the new monetary policy shock, which has standard deviation σ^{M}.[28] This is of course equivalent to adding a process of this form for targeted inflation, and combining it with a rule like equation (6) to ensure that realised inflation is always equal to targeted inflation.[29] We allow the shock $\varepsilon_t^{\mathrm{M}}$ to be correlated with the other structural shocks on the economy, to capture the central bank's endogenous reaction to such shocks. Allowing for correlation with all shocks is important to capture the possibility that the central bank may not be able to distinguish certain shocks. These correlations add six new parameters to estimate, and while we save some parameters from not having to estimate the rule, without further restrictions we would have three more parameters than in the original model. To make the modified model of comparable complexity to the original, we thus fix $\rho^{\mathrm{M}} = 0.99$, so the process well approximates an IMA(1,1), and we remove the MA components of the shocks to price and wage mark-ups, which were originally added to help generate MA behaviour in inflation. With these changes, both models have 36 estimated parameters.

In estimating our modified model, we broadly follow Smets & Wouters (2007), including using their original data.[30] However, we make three minor changes to the estimation method they used. For comparability, we also re-estimate the Smets & Wouters (2007) model with our modified procedure.

Firstly, we use uniform priors (from 0% to 100%) on all shock standard deviations, following Rabanal & Rubio-Ramírez (2005; 2008). With as many shocks as observables, non-uniform priors push estimates towards finding a role for all shocks, which is the opposite of what is needed if one wishes to tell a story of the sources of business cycles. They also make it much harder to compare predictive likelihoods across models, as they bias the variance of the predicted observations. Finally, non-uniform priors on shock standard deviations are particularly hard to interpret in the Smets & Wouters (2007) model, as all the shocks in that model are rescaled so they enter each equation

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[28] There is no offset in this equation, as in the Smets & Wouters (2007) model, an offset is added in the observation equation, which takes the form observed inflation = $\bar{\pi} + \pi_t$.

[29] The Fisher relationship in the Smets & Wouters (2007) model incorporates a shock ε_t^b, thus this shock also needs to appear in the monetary rule in order for the central bank to hit its desired inflation, much as shown in Section 2. We assume that the central bank has no difficulty in observing ε_t^b since there are as many shocks as observables implying shocks can be inferred from observables. The value of ϕ has no impact on equilibrium dynamics, providing $\phi > 1$. Thus, we do not attempt to estimate ϕ.

[30] We base our implementation of the model on the replication code provided by Johannes Pfeifer in https://github.com/JohannesPfeifer/DSGE_mod.

with a unit coefficient. Thus, the "structural" standard deviations are a complicated non-linear combination of many of the model's parameters.

Secondly, rather than initializing the state covariance matrix with an uncorrelated diffuse normal and one year of pre-sample data, we instead initialize it with the stationary distribution of the model and the entire 74 quarters of pre-sample data available in the Smets & Wouters (2007) datafile.[31] This ensures that the model's predictions have low variance even early on in the sample.

Finally, we correct a typo pointed out by Del Negro & Schorfheide (2012) which changes the coefficient multiplying the investment specific technological change in the evolution of the capital stock. Fixing this was particularly important here as the modified model ends up ascribing a large importance to investment specific technological change.

Table 1 shows the estimation results. For now, we just report posterior modes (TODO: credible sets). We have used diffuse uniform priors for all the new parameters. The results from re-estimating the original Smets & Wouters (2007) model are almost identical to those reported in Smets & Wouters (2007). However, the estimates for our modified model are drastically different, with much more flexible prices and wages, and lower habits and capital adjustment costs. This increased price flexibility is in line with some of the micro-data for posted prices (e.g. Bils & Klenow 2004; Klenow & Kryvtsov 2008).[32] Our estimates also feature larger shock standard deviations, and a much larger role for productivity shocks.[33] While the estimated shock standard deviations may seem very large in the modified model, the actual impacts of these shocks are far more modest,[34] as may be seen from the impulse responses shown in Appendix B. The largest impact on output to a one standard deviation impulse is from the total factor productivity shock, which raises output by about 0.7 percentage points.

[31] The datafile starts in 1947Q3, but Smets & Wouters (2007) only use the data from 1966Q1 to 2004Q4 for their main estimates. We use 1947Q3 to 1965Q4 as pre-sample data to initialize the state distribution, whereas Smets & Wouters (2007) use 1965Q1 to 1965Q4 for this purpose.

[32] If sales are excluded, then prices look much more persistent, but it is unclear why sales should be removed if attempting to calibrate a Calvo model.

[33] The total factor productivity shock explains 54% of the variance of output, and the investment specific shock explains a further 24%. The remainder is chiefly explained by the labour supply shock.

[34] This is partly an artefact of the rescaling of shocks used by Smets & Wouters (2007).

Variable	Prior Shape	Prior Mean	Prior SD	S&W Mode	S&W Re-estimated Mode	Modified Model Mode
φ	Normal	4.00	1.50	5.48	5.17	0.02
σ_c	Normal	1.50	0.38	1.39	1.42	1.23
λ	Beta	0.70	0.10	0.71	0.73	0.24
ξ_w	Beta	0.50	0.10	0.73	0.75	0.07
σ_l	Normal	2.00	0.75	1.92	2.01	3.32
ξ_p	Beta	0.50	0.10	0.65	0.64	0.13
ι_w	Beta	0.50	0.15	0.59	0.58	0.47
ι_p	Beta	0.50	0.15	0.22	0.25	0.18
ψ	Beta	0.50	0.15	0.54	0.40	0.90
Φ_p	Normal	1.25	0.13	1.61	1.55	1.36
r_π	Normal	1.50	0.25	2.03	2.04	
ρ	Beta	0.75	0.10	0.81	0.82	
r_y	Normal	0.12	0.05	0.08	0.12	
$r_{\Delta y}$	Normal	0.12	0.05	0.22	0.22	
$\bar{\pi}$	Gamma	0.62	0.10	0.81	0.65	0.61
$100(\beta^{-1}-1)$	Gamma	0.25	0.10	0.16	0.13	0.18
$\bar{l}$	Normal	0.00	2.00	-0.1	1.22	-1.06
$\bar{\gamma}$	Normal	0.40	0.10	0.43	0.51	0.32
α	Normal	0.30	0.05	0.19	0.19	0.29
σ_a	Uniform	50.0	28.87	0.45	0.47	0.51
σ_b	Uniform	50.0	28.87	0.24	0.23	0.14
σ_g	Uniform	50.0	28.87	0.52	0.52	0.55
σ_I	Uniform	50.0	28.87	0.45	0.45	31.91
σ_r	Uniform	50.0	28.87	0.24	0.24	
σ_p	Uniform	50.0	28.87	0.14	0.14	0.67
σ_w	Uniform	50.0	28.87	0.24	0.25	2.90
ρ_a	Beta	0.50	0.20	0.95	0.98	0.99
ρ_b	Beta	0.50	0.20	0.18	0.27	0.86
ρ_g	Beta	0.50	0.20	0.97	0.97	0.96
ρ_I	Beta	0.50	0.20	0.71	0.73	0.98
ρ_r	Beta	0.50	0.20	0.12	0.13	
ρ_p	Beta	0.50	0.20	0.90	0.99	0.91
ρ_w	Beta	0.50	0.20	0.97	0.97	0.97
μ_p	Beta	0.50	0.20	0.74	0.87	
μ_w	Beta	0.50	0.20	0.88	0.90	
ρ_{ga}	Normal	0.50	0.25	0.52	0.55	0.52
θ^{M}	Uniform	0.00	0.58			-0.37
σ^{M}	Uniform	50.0	28.87			0.29
$\mathrm{corr}(\varepsilon_t^{\mathrm{M}}, \varepsilon_t^{a})$	Uniform	0.00	0.58			-0.27
$\mathrm{corr}(\varepsilon_t^{\mathrm{M}}, \varepsilon_t^{b})$	Uniform	0.00	0.58			-0.46
$\mathrm{corr}(\varepsilon_t^{\mathrm{M}}, \varepsilon_t^{g})$	Uniform	0.00	0.58			0.02
$\mathrm{corr}(\varepsilon_t^{\mathrm{M}}, \varepsilon_t^{I})$	Uniform	0.00	0.58			0.10
$\mathrm{corr}(\varepsilon_t^{\mathrm{M}}, \varepsilon_t^{p})$	Uniform	0.00	0.58			0.13
$\mathrm{corr}(\varepsilon_t^{\mathrm{M}}, \varepsilon_t^{w})$	Uniform	0.00	0.58			-0.13

Table 1: Estimation results for the Smets & Wouters (2007) model and variant. The notation follows Smets & Wouters (2007).

The estimated process for inflation has a negative moving average component, in line both with the reduced form results of the prior empirical literature, and with the characterisation of optimal policy for a simple model that we gave in Section 3. Given the prior empirical literature's finding that IMA(1,1) or ARMA(1,1) models forecast very well, it should not be surprising that the modified model forecasts inflation better than the original one. Table 2 shows a reduction in one quarter ahead (in-sample) root mean squared error in inflation from 0.2899 to 0.2876. More surprising is that the modified model also forecasts nominal interest rates better than the original model, with the one quarter ahead (in-sample) root mean squared error in nominal interest rates reduced from 0.2442 to 0.2414. A standard Taylor rule does not appear to be necessary to explain nominal interest rate movements. If this model is true, the results of reduced-form empirical studies estimating Taylor rules may be driven by endogeneity and other identification failures.

Finally, Table 2 shows that the modified model attains a much higher likelihood for the observed inflation and nominal interest rate series.[35] Again, this is a shocking result. Apparently, the dynamics of nominal interest rates and inflation are worse explained by Smets & Wouters (2007) than by a model in which prices are almost flexible and in which inflation is entirely driven by the central bank's choices.

		Re-estimated S&W	Modified S&W
Root mean squared one quarter forecast error (in-sample)	Inflation	0.2899	0.2876
	Nominal interest rates	0.2442	0.2414
Likelihood of inflation and nominal interest rates		-51.57	-48.57

Table 2: Model performance for inflation and nominal interest rates, evaluated at the posterior mode of each specification.

Of course, this result should be taken with a pinch of salt. There are a variety of reasons why you may believe that existing behaviour of central bank governors is closer to that described by the original Smets & Wouters (2007) model than that described by our modified version. However, it is certainly suggestive of the possibility that central banks are targeting some path for inflation, ensuring that the causality in the Phillips curve runs from inflation to the output-gap, not in reverse.

Is the central bank behaving reasonably in our estimated modified version of the model? Using our estimates of the correlation between monetary policy shocks and

[35] It does not attain a higher likelihood overall. However, since the other observables are real, this suggests that the deficiency is on the real side of the economy.

the model's other structural shocks, we can infer the response of the central bank to each structural shock. I.e. we can calculate the parameters of the equation:

$$\varepsilon_t^{\mathrm{M}} = \theta_a \varepsilon_t^a + \theta_b \varepsilon_t^b + \theta_g \varepsilon_t^g + \theta_I \varepsilon_t^I + \theta_p \varepsilon_t^p + \theta_w \varepsilon_t^w + \hat{\varepsilon}_t^{\mathrm{M}},$$

where $\hat{\varepsilon}_t^{\mathrm{M}}$ is a "true" monetary policy shock, uncorrelated with all of the other structural shocks, and $\varepsilon_t^a, \varepsilon_t^b, \varepsilon_t^g, \varepsilon_t^I, \varepsilon_t^p, \varepsilon_t^w$ are the model's structural shocks (following the notation of Smets & Wouters (2007)). These parameters are as follows:

$$\theta_a = -0.15, \theta_b = -0.93, \theta_g = 0.0090, \theta_I = 0.00088, \theta_p = 0.054, \theta_w = -0.013.$$

Recall that the other rule parameters were:

$$\rho^{\mathrm{M}} = 0.99 \text{ (fixed)}, \qquad \theta^{\mathrm{M}} = -0.37.$$

To see if these parameters are reasonable, we turn off the $\hat{\varepsilon}_t^{\mathrm{M}}$ shock, then numerically optimise over $\rho^{\mathrm{M}}, \theta^{\mathrm{M}}, \varepsilon_t^a, \varepsilon_t^b, \varepsilon_t^g, \varepsilon_t^I, \varepsilon_t^p, \varepsilon_t^w$ to find the rule which minimises:

$$\sqrt{\mathbb{E}\left[\pi_t^2 + \frac{1}{16} x_t^2\right]}, \tag{8}$$

where (as ever) π_t is inflation relative to its target, and x_t is the output-gap, here defined following Smets & Wouters (2007). The value $\frac{1}{16}$ corresponds to an equal weight on annualized inflation and the output gap, broadly in line with the mandate of the US Federal Reserve.

The optimal parameters are:

$$\rho^{\mathrm{M}} = 0.26, \qquad \theta^{\mathrm{M}} = -0.38,$$

$$\theta_a = -0.0047, \theta_b = -0.00019, \theta_g = 0.0047, \theta_I = 9.26 \times 10^{-6}, \theta_p = 0.016, \theta_w = 0.0088.$$

These parameters reduce the objective (8) from 1.18 at the estimated parameters (but without the $\hat{\varepsilon}_t^{\mathrm{M}}$ shock) to 0.92. Thus, in standard deviation units, the estimated rule is about 22% worse. While a non-trivial gap, this at least suggests that the central bank is behaving broadly reasonably. Further suggestive evidence is provided by the fact that the signs of the optimal $\theta_{\cdot}$ parameters are identical to the estimated signs, in all cases except one.

If we allow slightly more complex rules, we can also improve upon the performance (in the sense of objective (8)) of the monetary rule from the original Smets & Wouters (2007) model, at our re-estimated parameters. This is a more difficult exercise than in our modified model as the relatively flat Phillips curve in the original Smets & Wouters (2007) model means even slight movements in inflation translate to large movements in the output gap. At the same time, the estimated model assigns an implausibly large role to inefficient mark-up shocks.[36] To see just how big this is, suppose prices were flexible so all mark-up movements were exogenous. Then the mean of the logarithm of gross mark-ups would be around 0.44, corresponding to a 55% mark-up. However, the standard deviation of the logarithm of gross mark-ups would be around 12.18![37] Despite this, we are able to attain a lower value for objective (8) if we let inflation follow a multiple-shock ARMA(1,2) process, with separate coefficients for lags 0, 1 and 2 of each of the model's structural shocks. While the original rule (but without the $\hat{\varepsilon}_t^{\mathrm{M}}$ shock) attains a value of 1.38 for objective (8), with the modified rule we are able to lower this to 1.30.

TODO WRITE UP MODEL ESTIMATED ON MONTHLY DATA

TODO MORE!

5 The zero lower bound

TODO (NOTE: Helps avoid the ZLB, longer rates are less likely to hit zero.)

6 Conclusion

TODO

7 References

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[36] Other authors have noted this point. For example, Chari, Kehoe & McGrattan (2009) report that with the original Smets & Wouters (2007) estimates, the standard deviation of the wage mark-up is 2587%.

[37] Authors calculation. Yet more outrageously, plugging these numbers into the standard formulas for the moments of a log-normal distribution gives a mean of 2.6×10^{34}% and a standard deviation of 4.3×10^{66}%. Of course, the first order approximation taken to the model omits the effects of the variance on the mean, so perhaps this is an unfair exercise.

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Appendix A Proofs and supplemental results

A.1 Phillips curve based forecasting with ARMA(1,1) policy shocks

As before, we have the monetary rule:

$$i_t = r_t + \phi\pi_t + \zeta_t,$$

which combined with the Fisher equation gives:

$$\mathbb{E}_t\pi_{t+1} = \phi\pi_t + \zeta_t.$$

Suppose ζ_t follows the ARMA(1,1) process:

$$\zeta_t = \rho\zeta_{t-1} + \varepsilon_{\zeta,t} + \theta\varepsilon_{\zeta,t-1},$$

with $\rho, \theta \in (-1,1)$. Then from matching coefficients, with $\phi > 1$ we have the unique solution:

$$\pi_t = -\frac{1}{\phi-\rho}\left[\zeta_t + \frac{\theta}{\phi}\varepsilon_{\zeta,t}\right].$$

Thus:

$$\pi_t - \rho\pi_{t-1} = -\frac{1}{\phi-\rho}\left(1+\frac{\theta}{\phi}\right)\left[\varepsilon_{\zeta,t} + \frac{\phi-\rho}{\phi+\theta}\theta\varepsilon_{\zeta,t-1}\right],$$

so π_t also follows an ARMA(1,1) process. Suppose for now that $-\rho \le \theta$, which is likely to be satisfied in reality as we expect ρ to be large and positive, while θ should be close to zero. (For example, Dotsey, Fujita & Stark (2018) find that an IMA(1,1) model fits inflation well, in which case $-\rho = -1 < \theta$ as required.) Then $0 < \frac{\phi-\rho}{\phi+\theta} < 1$, so $\left|\frac{\phi-\rho}{\phi+\theta}\theta\right| < 1$ meaning the process for inflation is invertible. With inflation following an invertible linear process, the full-information optimal forecast of π_{t+1} is a linear combination of $\pi_t, \pi_{t-1}, \ldots$. In particular, as before x_t is not useful.

In the unlikely case in which $-\rho > \theta$, of if the forecaster's information set $\mathcal{I}_t$ is smaller than $\{\pi_t, x_t, \pi_{t-1}, x_{t-1}, \ldots\}$,[38] then x_t may contain some useful information. Combining the solution for inflation with the Phillips curve:

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa x_t + \kappa\omega_t,$$

gives:

$$x_t = -\frac{1}{\kappa}\left[\frac{1-\beta\rho}{\phi-\rho}\left(\zeta_t + \frac{\theta}{\phi}\varepsilon_{\zeta,t}\right) - \beta\frac{\theta}{\phi}\varepsilon_{\zeta,t}\right] - \omega_t = \frac{1}{\kappa}\left[(1-\beta\rho)\pi_t + \beta\frac{\theta}{\phi}\varepsilon_{\zeta,t}\right] - \omega_t.$$

[38] We nonetheless assume that π_t and x_t are in $\mathcal{I}_t$.

In this case, it is possible that $\mathbb{E}[\pi_{t+1}|\mathcal{I}_t] \neq \mathbb{E}[\pi_{t+1}|\mathcal{J}_t]$ as x_t provides an independent signal about $\varepsilon_{\zeta,t}$.

There are two important special cases. If $\omega_t = 0$, and the forecaster knows this, then:

$$\varepsilon_{\zeta,t} = \frac{\phi}{\beta\theta}[\kappa x_t - (1-\beta\rho)\pi_t],$$

so:

$$\zeta_t = -\left(\phi - \frac{1}{\beta}\right)\pi_t - \frac{\kappa}{\beta}x_t,$$

which enables the forecaster to form the full-information optimal forecast:

$$\mathbb{E}_t\pi_{t+1} = -\frac{1}{\phi-\rho}\left(\rho\zeta_t + \theta\varepsilon_{\zeta,t}\right) = \frac{1}{\beta}(\pi_t - \kappa x_t).$$

(This formula also follows immediately from the Phillips curve.) Note that the output gap has what Dotsey, Fujita & Stark (2018) call the "wrong" sign, meaning Phillips curve based forecasting regressions may have surprising results. However, in the general case in which ω_t has positive variance, then output's signal about $\varepsilon_{\zeta,t}$ will be polluted by the noise from ω_t, making it much less informative. Indeed, with ϕ large, as we expect, then $\frac{\theta}{\phi}\varepsilon_{\zeta,t}$ will have low variance, making it more likely that it is drowned out by the noise from ω_t.

The second important special case is when $\varepsilon_{\zeta,t} = 0$, and again the forecaster knows this. In this case, much as in the main text:

$$\mathbb{E}_t\pi_{t+1} = \rho\pi_t - \frac{1}{\phi-\rho}\left(1+\frac{\theta}{\phi}\right)\left[\mathbb{E}_t\varepsilon_{\zeta,t+1} + \frac{\phi-\rho}{\phi+\theta}\theta\varepsilon_{\zeta,t}\right] = \rho\pi_t,$$

so x_t is unhelpful.

The general case will inherit aspects of these two special cases, as well as the case in which π_t's stochastic process was invertible. Inflation and its lags will certainly help forecast inflation, but the output gap may also provide a little extra information, possibly with the "wrong" sign.

A.2 Robustness to non-unit responses to real interest rates

Suppose that the central bank is unable to respond with a precise unit coefficient to real interest rates, so instead follows the monetary rule:

$$i_t = (1+\gamma)r_t + \phi\pi_t + \zeta_t,$$

where $\gamma \in \mathbb{R}$ is some small value giving the departure from unit responses.

For simplicity, suppose the rest of the model takes the same form as in Section 1, with:

$$\begin{aligned} x_t &= \delta \mathbb{E}_t x_{t+1} - \varsigma(r_t - n_t), \\ \pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \kappa \omega_t, \\ i_t &= r_t + \mathbb{E}_t \pi_{t+1}. \end{aligned}$$

We suppose $\phi > 1$, but do not make any assumptions on the signs of $\delta, \beta, \kappa, \varsigma, \gamma$, beyond assuming that $\varsigma \neq 0$ (so monetary policy has some effect on the output gap) and $\kappa \neq 0$ (so monetary policy has some effect on inflation, via the output gap).

Combining the monetary rule with the Fisher equation gives:

$$\mathbb{E}_t \pi_{t+1} = \gamma r_t + \phi \pi_t + \zeta_t,$$

so:

$$r_t = \frac{1}{\gamma}\left(\mathbb{E}_t \pi_{t+1} - \phi \pi_t - \zeta_t\right),$$

meaning:

$$x_t = \delta \mathbb{E}_t x_{t+1} - \frac{\varsigma}{\gamma}\left(\mathbb{E}_t \pi_{t+1} - \phi \pi_t\right) + \varsigma n_t + \frac{\varsigma}{\gamma}\zeta_t.$$

Then, since:

$$\mathbb{E}_t \pi_{t+1} = \frac{1}{\beta}\pi_t - \frac{\kappa}{\beta}x_t - \frac{\kappa}{\beta}\omega_t,$$

We have that:

$$\mathbb{E}_t x_{t+1} = \left(\frac{1}{\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}\right)x_t - \frac{\varsigma}{\delta\gamma}\left(\phi - \frac{1}{\beta}\right)\pi_t - \frac{\varsigma}{\delta\gamma}\left(\gamma n_t + \zeta_t + \frac{\kappa}{\beta}\omega_t\right).$$

Woodford (2003) (Addendum to Chapter 4, Proposition C.1) proves that this model is determinate if and only if both eigenvalues of the matrix:

$$M := \begin{bmatrix} \frac{1}{\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta} & -\frac{\varsigma}{\delta\gamma}\left(\phi - \frac{1}{\beta}\right) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

are outside of the unit circle, which in turn is proven to hold if and only if *EITHER*: Case I: $1 < \det M$, $0 < 1 + \det M - \operatorname{tr} M$, and $0 < 1 + \det M + \operatorname{tr} M$, *OR* Case II: $0 > 1 + \det M - \operatorname{tr} M$, and $0 > 1 + \det M + \operatorname{tr} M$. Note:

$$\det M = \frac{1}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}\phi,$$

$$\operatorname{tr} M = \frac{1}{\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta} + \frac{1}{\beta}.$$

Thus, Case I requires:

$$1 < \det M = \frac{1}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}\phi,$$

$$0 < 1 + \det M - \operatorname{tr} M = \frac{(1-\beta)(1-\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(\phi - 1),$$

$$\text{and } 0 < 1 + \det M + \operatorname{tr} M = \frac{(1+\beta)(1+\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(1 + \phi).$$

And Case II requires:

$$0 > 1 + \det M - \operatorname{tr} M = \frac{(1-\beta)(1-\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(\phi - 1),$$

$$\text{and } 0 > 1 + \det M + \operatorname{tr} M = \frac{(1+\beta)(1+\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(1 + \phi).$$

To see when these conditions are satisfied, first suppose that $\frac{\varsigma\kappa}{\gamma\beta\delta} < 0$, so $\frac{\varsigma\kappa}{\gamma\beta\delta} = -\frac{|\varsigma\kappa|}{|\gamma||\beta\delta|}$. Then if γ is sufficiently small in magnitude, it is immediately clear that all three conditions of Case I are satisfied, since $\phi > 0$, $\phi - 1 > 0$ and $1 + \phi > 0$. In particular, in this case we need:

$$|\gamma| < |\varsigma\kappa| \min \left\{ \begin{array}{c} \dfrac{\phi}{\max\{0, -(\operatorname{sign}(\beta\delta) - |\beta\delta|)\}}, \\ \dfrac{\phi - 1}{\max\{0, -(\operatorname{sign}(\beta\delta))(1-\beta)(1-\delta)\}}, \\ \dfrac{1 + \phi}{\max\{0, -(\operatorname{sign}(\beta\delta))(1+\beta)(1+\delta)\}} \end{array} \right\}.$$

Alternatively, suppose that $\frac{\varsigma\kappa}{\gamma\beta\delta} > 0$, so $\frac{\varsigma\kappa}{\gamma\beta\delta} = \frac{|\varsigma\kappa|}{|\gamma||\beta\delta|}$. Then, similarly, if γ is sufficiently small in magnitude, both conditions of Case II are satisfied, since $\phi - 1 > 0$ and $1 + \phi > 0$. In particular, in this case we need:

$$|\gamma| < |\varsigma\kappa| \min \left\{ \begin{array}{c} \dfrac{\phi - 1}{\max\{0, (\operatorname{sign}(\beta\delta))(1-\beta)(1-\delta)\}}, \\ \dfrac{1 + \phi}{\max\{0, (\operatorname{sign}(\beta\delta))(1+\beta)(1+\delta)\}} \end{array} \right\}.$$

Thus, it is always sufficient for determinacy that:

$$|\gamma| < |\varsigma\kappa| \min \left\{ \begin{matrix} \dfrac{\phi}{\max\{0, -(\operatorname{sign}(\beta\delta) - |\beta\delta|)\}}, \\ \dfrac{\phi - 1}{|(1-\beta)(1-\delta)|}, \\ \dfrac{1+\phi}{|(1+\beta)(1+\delta)|} \end{matrix} \right\}.$$

Since the right hand side is strictly positive, there is a positive measure of γ for which we have determinacy.

A.3 If inflation is identical, other endogenous variables are identical

Let x_t and $\tilde{x}_t$ be vectors stacking the endogenous variables other than inflation in the economy with our rule and the economy with the alternative rule, respectively. We assume without loss of generality that they are all zero in steady state. By linearity, the equations other than the monetary rule or monetary policy first order condition must have the form:

$$0 = \mathcal{A}x_{t-1} + a\pi_{t-1} + \mathcal{B}x_t + b\pi_t + \mathcal{C}\mathbb{E}x_{t+1} + c\mathbb{E}\pi_{t+1} + \sum_{n=1}^{N} d_n\varepsilon_{n,t}, \tag{9}$$

in the economy with our rule, and they must have the form:

$$0 = \mathcal{A}\tilde{x}_{t-1} + a\tilde{\pi}_{t-1} + \mathcal{B}\tilde{x}_t + b\tilde{\pi}_t + \mathcal{C}\mathbb{E}\tilde{x}_{t+1} + c\mathbb{E}\tilde{\pi}_{t+1} + \sum_{n=1}^{N} d_n\varepsilon_{n,t},$$

in the economy with the alternative rule. (Here, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are square matrices, while a, b and c are scalars, and $d_1, \dots, d_N$ are vectors.) Since $\pi_t \equiv \tilde{\pi}_t$, $x_t \equiv \tilde{x}_t$ must solve equation (9). It will be the unique solution providing the model has no source of indeterminacy other than perhaps monetary policy. For example, in a three equation NK model, given that $\pi_t \equiv \tilde{\pi}_t$, the Phillips curve implies that the output gap must agree in the two economies, thus the Euler equation then implies that the interest rate must also agree.

A.4 Solution properties of first welfare example

Recall, that for $k > 1$ the solution must satisfy the recurrence relation:

$$\theta_k + \frac{\lambda}{\kappa^2}(\theta_k - \beta\theta_{k+1}) - \beta\frac{\lambda}{\kappa^2}(\theta_{k-1} - \beta\theta_k) = 0.$$

The characteristic equation of this recurrence relationship has roots:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)\pm\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)^2-\left(2\beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}$$
$$=\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)\pm\sqrt{\left(1+(1+\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}.$$

The positive root satisfies:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)+\sqrt{\left(1+(1+\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}$$
$$>\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)+\sqrt{\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}$$
$$=\frac{1+\frac{\lambda}{\kappa^2}-\beta(1-\beta)\frac{\lambda}{\kappa^2}}{\beta\frac{\lambda}{\kappa^2}}>\frac{1+\frac{\lambda}{\kappa^2}-(1-\beta)\frac{\lambda}{\kappa^2}}{\beta\frac{\lambda}{\kappa^2}}=1+\frac{1}{\beta\frac{\lambda}{\kappa^2}}>1.$$

The negative root satisfies:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)^2-\left(2\beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}$$
$$>\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}=0,$$

and:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+(1+\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}$$
$$<\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}=1.$$

Hence, the positive root is greater than 1, while the negative root is in $(0,1)$. Thus for $k \geq 1$:

$$\theta_k = \theta_1 \left[\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta^2 \frac{\lambda}{\kappa^2}\right) - \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta^2 \frac{\lambda}{\kappa^2}\right)^2 - \left(2\beta \frac{\lambda}{\kappa^2}\right)^2}}{2\beta \frac{\lambda}{\kappa^2}} \right]^{k-1}.$$

Hence, θ_0, θ_1 and θ_2 are the unique solution of the three linear (in θ_0, θ_1 and θ_2) equations:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\theta_1 + \frac{\lambda}{\kappa^2}(\theta_1 - \beta\theta_2) - \beta\frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\theta_2 = \theta_1 \left[\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta^2 \frac{\lambda}{\kappa^2}\right) - \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta^2 \frac{\lambda}{\kappa^2}\right)^2 - \left(2\beta \frac{\lambda}{\kappa^2}\right)^2}}{2\beta \frac{\lambda}{\kappa^2}} \right].$$

A.5 Solution under discretion of first welfare example

Under discretion, we have the standard first order condition:

$$\pi_t + \frac{\lambda}{\kappa} x_t = 0,$$

i.e.:

$$\kappa \sum_{k=0}^{\infty} \theta_k \omega_{t-k} + \frac{\lambda}{\kappa} \sum_{k=0}^{\infty} (\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])\omega_{t-k} = 0,$$

so:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\forall k \geq 1, \qquad \theta_k + \frac{\lambda}{\kappa^2}(\theta_k - \beta\theta_{k+1}) = 0.$$

The latter recurrence relation has the general solution $\theta_k = \theta_1 \left(\frac{\kappa^2}{\beta\lambda} + \frac{1}{\beta}\right)^{k-1}$, which is explosive as $\beta < 1$. Thus, we must have $\theta_1 = \theta_2 = \cdots = 0$. Hence, $\theta_0 = \frac{\lambda}{\lambda + \kappa^2}$.

A.6 Solution under the timeless perspective of first welfare example

The timeless perspective (Woodford 1999) leads to the first order condition:

$$\pi_t + \frac{\lambda}{\kappa}(x_t - x_{t-1}) = 0,$$

i.e.:

$$\kappa \sum_{k=0}^{\infty} \theta_k \omega_{t-k} + \frac{\lambda}{\kappa}\left[\sum_{k=0}^{\infty}(\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])\omega_{t-k} - \sum_{k=1}^{\infty}(\theta_{k-1} - \beta\theta_k - \mathbb{1}[k-1=0])\omega_{t-k}\right] = 0,$$

so:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\theta_1 + \frac{\lambda}{\kappa^2}(\theta_1 - \beta\theta_2) - \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$

$$\forall k > 1, \qquad \theta_k + \frac{\lambda}{\kappa^2}(\theta_k - \beta\theta_{k+1}) - \frac{\lambda}{\kappa^2}(\theta_{k-1} - \beta\theta_k) = 0.$$

The roots of the characteristic equation corresponding to the latter recurrence relation are:

$$\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) \pm \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2 - 4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}.$$

The positive root satisfies:

$$\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) + \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2 - 4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}} > \frac{\frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}}{2\beta\frac{\lambda}{\kappa^2}} = \frac{1+\beta}{2\beta} > 1.$$

The negative root satisfies:

$$\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) - \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2 - 4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}} > \frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) - \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}} = 0,$$

and:

$$\begin{aligned}
&\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)^2-4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}\\
&\qquad=\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{1+(1-\beta)^2\left(\frac{\lambda}{\kappa^2}\right)^2+2(1+\beta)\frac{\lambda}{\kappa^2}}}{2\beta\frac{\lambda}{\kappa^2}}\\
&\qquad<\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{1+(1-\beta)^2\left(\frac{\lambda}{\kappa^2}\right)^2+2(1-\beta)\frac{\lambda}{\kappa^2}}}{2\beta\frac{\lambda}{\kappa^2}}\\
&\qquad=\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+(1-\beta)\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}=\frac{2\beta\frac{\lambda}{\kappa^2}}{2\beta\frac{\lambda}{\kappa^2}}=1.
\end{aligned}$$

Hence, the positive root is greater than 1, while the negative root is in $(0,1)$. Thus for $k \geq 1$:

$$\theta_k=\theta_1\left[\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)^2-4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}\right]^{k-1}.$$

Hence, θ_0, θ_1 and θ_2 are the unique solution of the three linear (in θ_0, θ_1 and θ_2) equations:

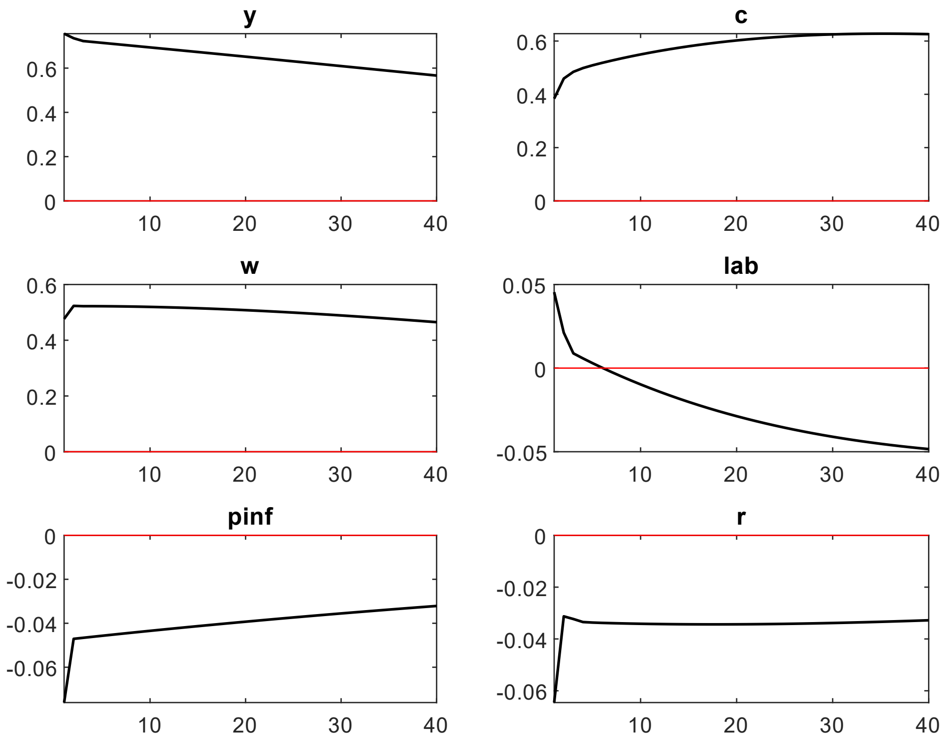
$$\theta_0+\frac{\lambda}{\kappa^2}(\theta_0-\beta\theta_1-1)=0,$$

$$\theta_1+\frac{\lambda}{\kappa^2}(\theta_1-\beta\theta_2)-\frac{\lambda}{\kappa^2}(\theta_0-\beta\theta_1-1)=0,$$

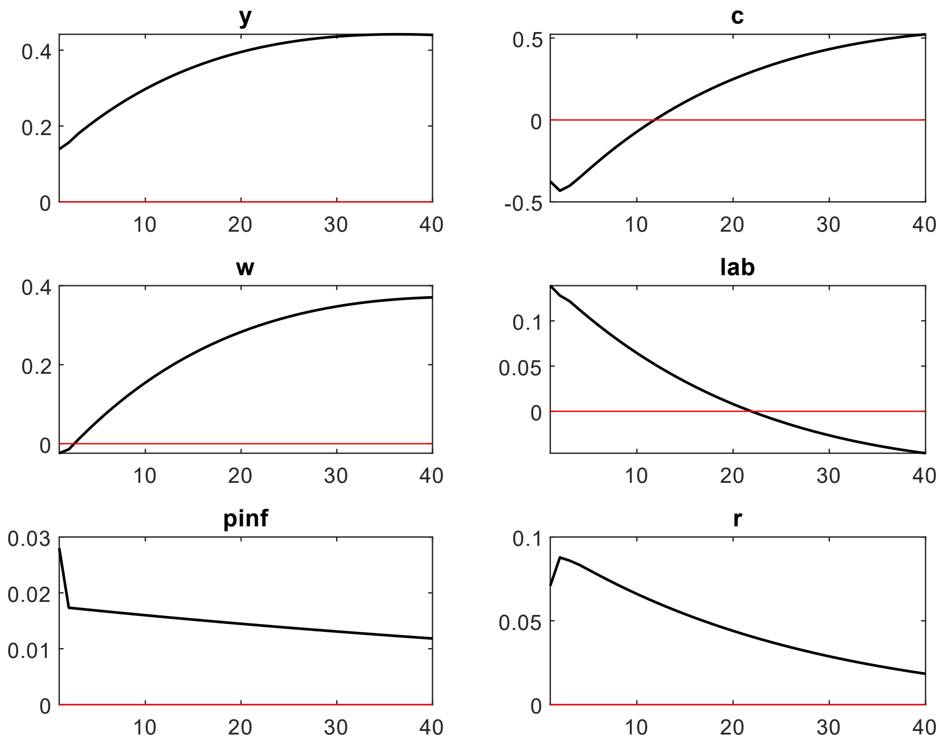
$$\theta_2=\theta_1\left[\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)^2-4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}\right].$$

Appendix B Impulse responses in the modified Smets & Wouters (2007) model

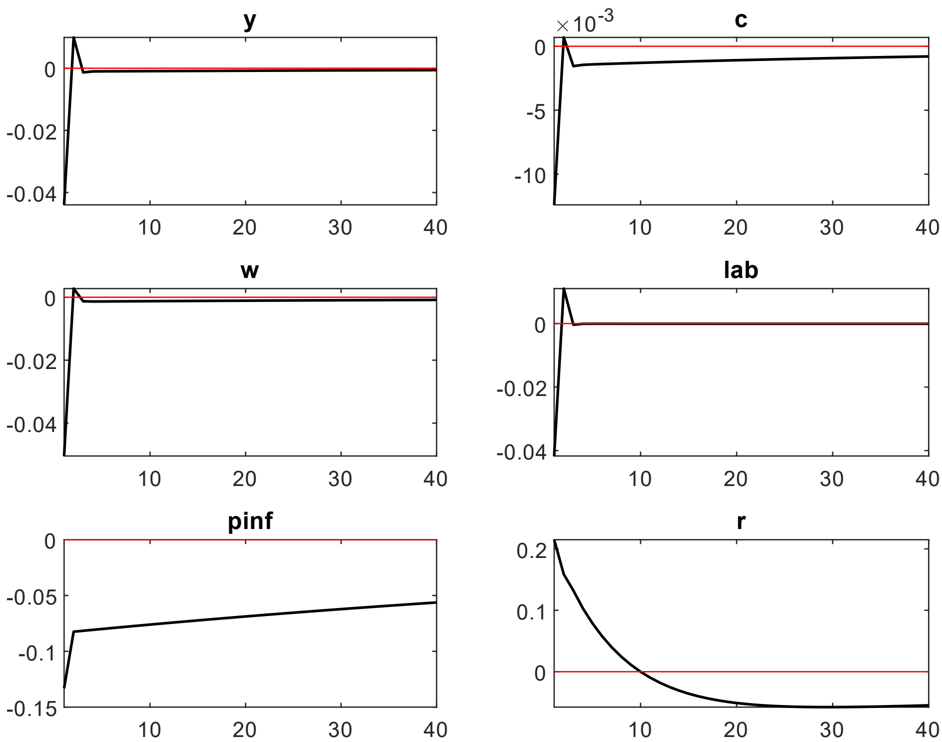
In each of the figures below, "y" refers to output, "c" refers to consumption, "w" refers to real wages, "lab" refers to total hours worked, "pinf" refers to inflation and "r" refers to nominal interest rates. All graphs are in percentage points. Since the driving shocks are correlated, to produce impulse responses we need to take a stance on which shocks respond contemporaneously to which other shocks. In line with the story of our paper, we suppose that monetary policy reacts contemporaneously to all of the other shocks in the model, while the other shocks do not respond to monetary policy. I.e. we order the monetary policy shock last before taking the Cholesky decomposition.



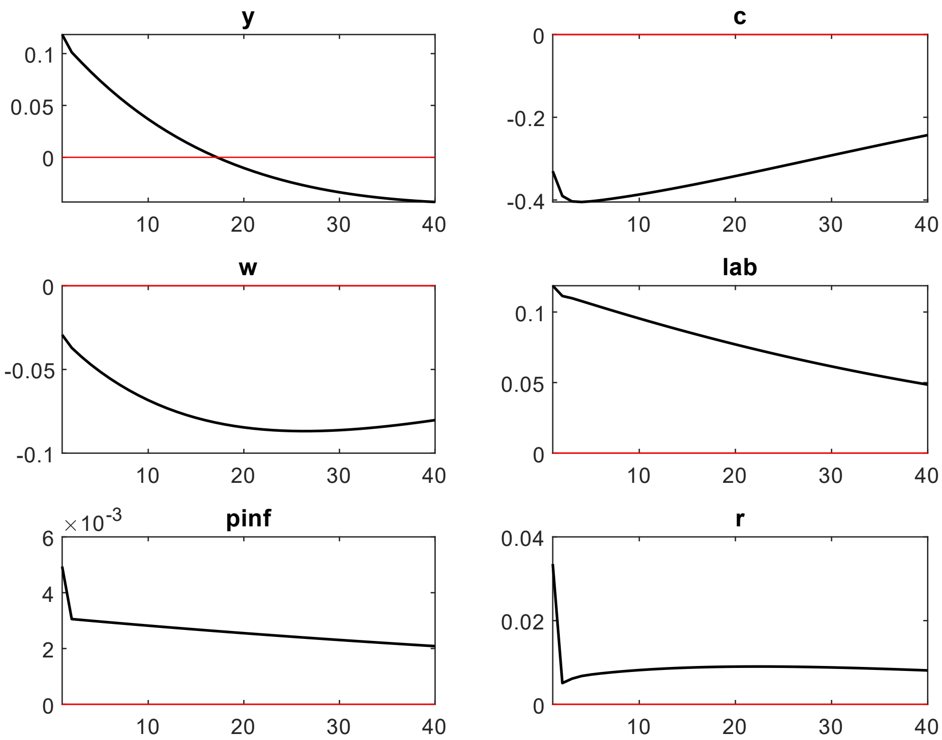
Response to a total factor productivity shock



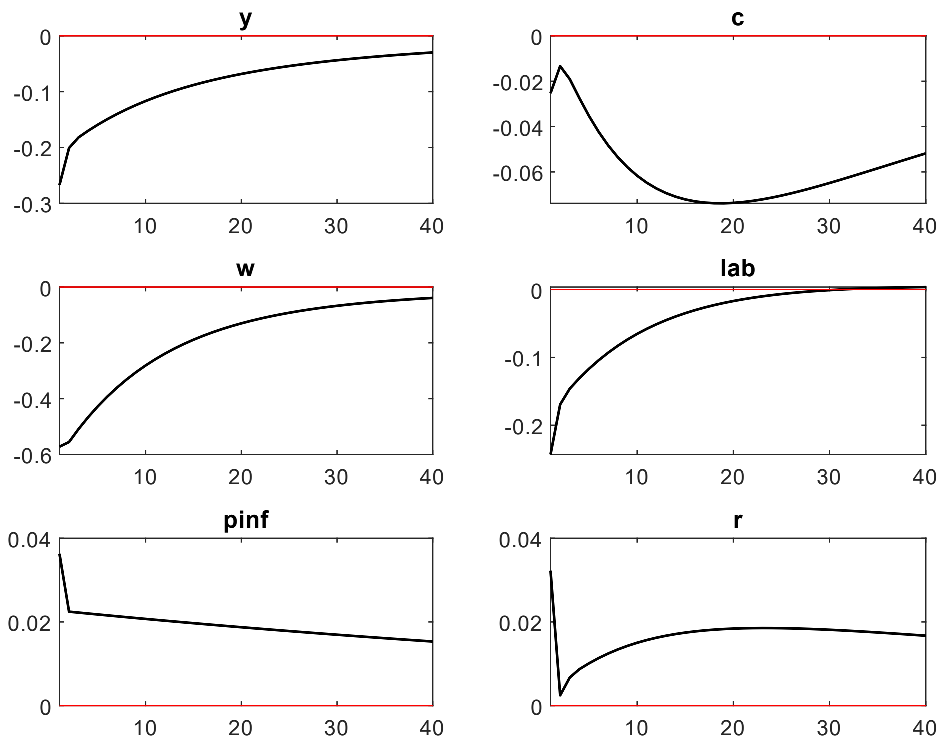
Response to an investment specific technology shock



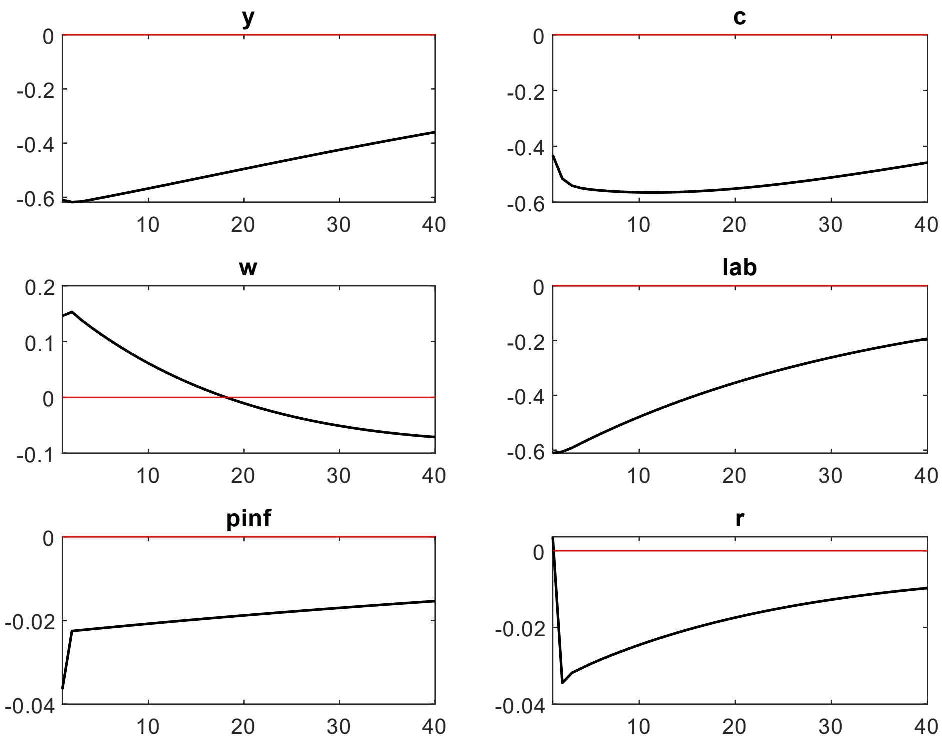
Response to a risk premium shock



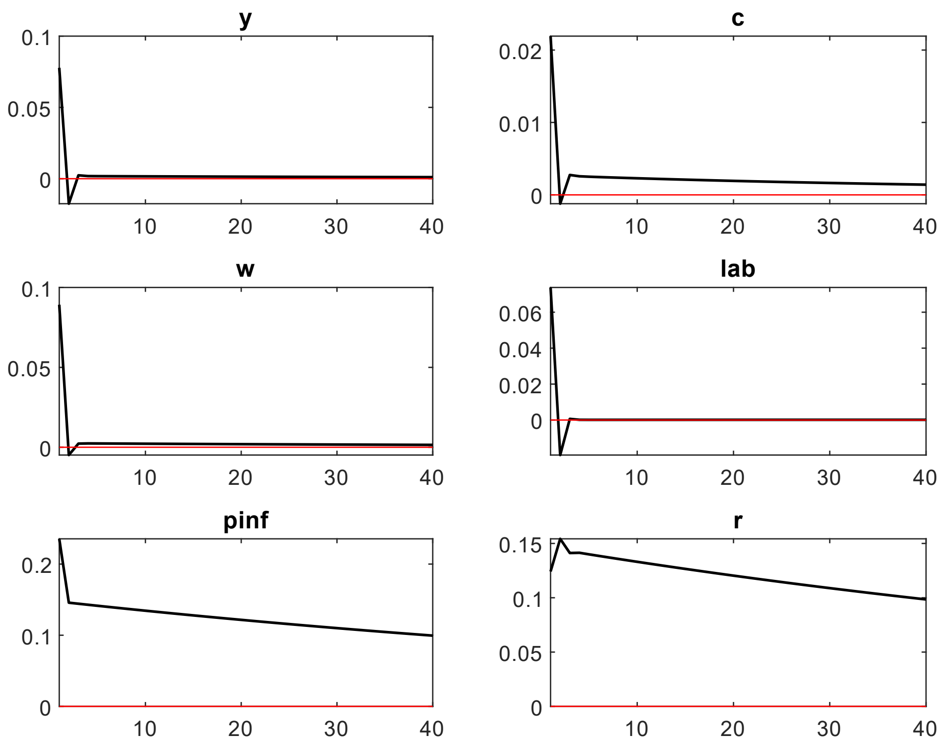
Response to an exogenous spending shock



Response to a price mark-up shock



Response to a wage mark-up shock



Response to a monetary policy shock