PhD Macro 2 Topic 7 Questions

1) Solve the non-stochastic asset eating problem with $u(x,c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, and $f(x,c) = rx - c$ using a Hamiltonian.

2) Suppose an agent maximises $\int_0^\infty e^{-\rho t} \log c(t)\, dt$ subject to $\dot{x}(t) = rx(t) + y(t) - c(t)$ where $y(t)$ is an exogenous function. Derive the solution for $c(t)$ and $x(t)$.

3) Suppose an agent maximises:

$$\psi x(T) + \int_0^T \left(-\frac{1}{2}\left(\bar{c} - c(t)\right)^2 + \phi x(t) - \frac{1}{2}\theta \dot{x}(t)^2 \right) dt$$

subject to $\dot{x}(t) = rx(t) - c(t)$ and $x(0) = x_0$. Derive the solution for $c(t)$ and $x(t)$. *Hint: Conjecture a solution to the FOCs that is linear in* e^{-rt} *and* e^{rt}.

4) Derive the first order and transversality conditions for the Ramsey-Cass-Koopmans model of exogenous growth with the CES production function:

$$Y(t) = Y(0)\left[\alpha \left(\frac{K(t)}{K(0)}\right)^{\frac{\zeta-1}{\zeta}} + (1-\alpha)\left(\frac{A(t)L(t)}{A(0)L(0)}\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta}{\zeta-1}},$$

where $L(t) = L_0 e^{nt}$ and $A(t) = A_0 e^{gt}$. Find the steady-state, and analyse the stability around that point.

5) [From the 2017 Resit] Suppose that a monopolist owns the entire global oil stock. Let $x(t)$ be the amount of oil remaining in the ground at t, where $x(0) = x_0 > 0$. Assume that the unit cost of resource extraction is constant at $c > 0$, and that the global demand for oil at a price p, at time t, with remaining oil stock x, is given by:

$$q(t,x,p) = \frac{k}{\left[p\left(1+\tau(t,x)\right)\right]^2},$$

where $\tau(t,x)$ is an exogenously given tax on purchasing oil, paid by consumers, not by the monopolist and where $k > 0$. Suppose further that the monopolist maximises the present value of their profits, given a constant real interest rate r, i.e.

$$\int_0^\infty e^{-rt} q\big(t, x(t), p(t)\big)[p(t) - c]\, dt,$$

subject to the resource constraint:

$$\dot{x}(t) = -q\big(t, x(t), p(t)\big).$$

a) Introducing the co-state, $\lambda(t)$, form the current value Hamiltonian for the monopolist's problem, and use it to derive the first order and transversality conditions for an optimum. Simplify one of these conditions to derive an expression for the price in terms of c and λ.

b) Now, suppose that in fact $\tau(t,x) = \sqrt{1+\gamma t} - 1$, for all t, x, where $\tau_1 > 0$.

i. Derive a closed form expression for λ in terms of $\lambda(0)$ and r, and use it to simplify the transversality constraint. Interpret the result.

ii. Show that:

$$x(t) = x_0 - \int_0^{\tau} \frac{k}{4(c+\lambda(0)e^{r\tau})^2(1+\gamma\tau)} d\tau$$

iii. Hence, show that were it the case that $\lambda(0) = 0$, then it would be the case that:

$$x(t) = x_0 - \frac{k}{4\gamma c^2}\log(1+\gamma t).$$

iv. Without any direct appeal to the transversality condition, explain informally why:

$$x(t) = \max\left\{0, x_0 - \frac{k}{4\gamma c^2}\log(1+\gamma t)\right\},$$

cannot be an optimal solution.

v. Qualitatively describe the properties of the optimal solution.

6) Solve the non-stochastic asset eating problem with $u(x,c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, and $f(x,c) = rx - c$ using the HJB equation.

7) Recall that in the stochastic asset eating model the value of x was a function of $\frac{\langle 1_n, 1_n\rangle\langle r, r\rangle - (\langle 1_n, r\rangle - \sigma)^2}{2\sigma\rho\langle 1_n, 1_n\rangle}$, where $\langle u, v\rangle \coloneqq u'(\Lambda\Lambda')^{-1}v$. What is the value of this quantity in the $n = 2$ case as a function of the elements of $\Sigma \coloneqq \Lambda\Lambda'$? For the value function to remain finite, what must be true or r as the two assets tend towards perfect correlation (i.e. $\det\Sigma \to 0$)? What happens in the case when the first asset is riskless (i.e. $\Lambda_{11} = \Lambda_{12} = 0$)?

8) Solve the stochastic asset eating model in the log-utility special case and verify that the b you find coincides with the one we obtained by taking the limit as $\sigma \to 1$.

9) Complete the following exercises from "Introduction to Modern Economic Growth" (Acemoglu):

a)

EXERCISE 7.4. This exercise asks you to use the Euler-Legrange equation [redacted] to solve the canonical problem that motivated Euler and Legrange, that of finding the shortest distance between two points in a plane. In particular, consider a two dimensional plane and two points on this plane with coordinates (z_0, u_0) and (z_1, u_1). We would like to find the curve that has the shortest length that connects these two points. Such a curve can be represented by a function $x : \mathbb{R} \to \mathbb{R}$ such that $u = x(z)$, together with initial and terminal conditions $u_0 = x(z_0)$ and $u_1 = x(z_1)$. It is also natural to impose that this curve $u = x(z)$ be smooth, which corresponds to requiring that the solution be continuously differentiable so that $x'(z)$ exists.

To solve this problem, observe that the (arc) length along the curve x can be represented as

$$A[x(z)] \equiv \int_{z_1}^{z_2} \sqrt{1 + [x'(z)]^2} dz.$$

The problem is to minimize this object by choosing $x(z)$.

Now, without loss of any generality let us take $(z_0, u_0) = (0, 0)$ and let $t = z$ to transform the problem into a more familiar form, which becomes that of maximizing

$$-\int_0^{t_1} \sqrt{1 + [x'(t)]^2} dt.$$

Prove that the solution to this problem requires

$$\frac{d\left[x'(t)\left(1 + (x'(t))^2\right)\right]}{dt} = 0.$$

Show that this is only possible if $x''(t) = 0$, so that the shortest path between two points is a straight-line.

b)

EXERCISE 7.16. * Consider the following maximization problem:

$$\max_{x(t), y(t)} -\int_0^1 x(t)^2 dt$$

subject to

$$\dot{x}(t) = y(t)^2$$

$x(0) = 0$ and $x(1) = 1$, where $y(t) \in \mathbb{R}$. Show that there does not exist a continuously differentiable solution to this problem.

c)

EXERCISE 7.18. Consider the following optimal growth model without discounting:

$$\max \int_0^\infty \left[u\left(c\left(t\right)\right) - u\left(c^*\right)\right] dt$$

subject to

$$\dot{k}\left(t\right) = f\left(k\left(t\right)\right) - c\left(t\right) - \delta k\left(t\right)$$

with initial condition $k\left(0\right) > 0$, and c^* defined as the golden rule consumption level

$$c^* = f\left(k^*\right) - \delta k^*$$

where k^* is the golden rule capital-labor ratio given by $f'\left(k^*\right) = \delta$.

(1) Set up the Hamiltonian for this problem with costate variable $\lambda\left(t\right)$.

(2) Characterize the solution to this optimal growth program.

(3) Show that the standard transversality condition that $\lim_{t\to\infty} \lambda\left(t\right) k\left(t\right) = 0$ is not satisfied at the optimal solution. Explain why this is the case.