PhD Macro 2 Topic 5 Questions

1) Suppose we observe $x_1, \dots, x_T$, where:

$$x_t = \rho x_{t-1} + \varepsilon_t, \;\; \varepsilon_t \sim \mathrm{NIID}(0, \sigma^2), \;\; x_0 = m_0,$$

where m_0 is an **known** parameter representing the point at which the process was started. Derive closed-form expressions for the maximum likelihood estimates of ρ and σ. Show that the maximum likelihood estimate of ρ is identical to the results of an OLS regression of $x_2, \dots, x_T$ on $x_1, \dots, x_{T-1}$.

2) Suppose we observe $x_1, \dots, x_T$, where:

$$x_t = \rho x_{t-1} + \varepsilon_t, \;\; \varepsilon_t \sim \mathrm{NIID}(0, \sigma^2), \;\; x_0 = m_0,$$

where m_0 is an **unknown** parameter representing the point at which the process was started. Derive closed-form expressions for the maximum likelihood estimates of ρ, σ and m_0. Show that the maximum likelihood estimate of ρ is asymptotically equivalent to the results of an OLS regression of $x_2, \dots, x_T$ on $x_1, \dots, x_{T-1}$. Is the maximum likelihood estimator of m_0 consistent?

3) Prove the “useful property of the Normal distribution” from the slides in the case when U and V are scalars.

4) [Part of a 2016 exam question.] Suppose that:

$$z_t = z_{t-1} + \sigma_z \varepsilon_{z,t}$$

$$a_t = z_t + \sigma_a \varepsilon_{a,t}$$

where $\varepsilon_{z,t}, \varepsilon_{a,t} \sim \mathrm{NIID}(0,1)$ (i.e. they are independent, identically distributed standard normals), and where $\sigma_z > 0$ and $\sigma_a > 0$. Let $\mathcal{F}_t \coloneqq [a_t, a_{t-1}, \dots]$ be all the available information in period t. I.e. a_t is observable, but z_t is not.

Suppose that $z_t | \mathcal{F}_t \sim \mathrm{N}(m_t, v_t)$.

(a) Prove that $z_{t+1} | \mathcal{F}_{t+1} \sim \mathrm{N}(m_{t+1}, v_{t+1})$, fully explaining your steps, and giving simple expressions for m_{t+1} and v_{t+1} in terms of m_t, v_t, a_{t+1}, σ_z and σ_a. You may assume that if $\begin{bmatrix} U \\ V \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} \mu_U \\ \mu_V \end{bmatrix}, \begin{bmatrix} \Sigma_{UU} & \Sigma_{UV} \\ \Sigma_{VU} & \Sigma_{VV} \end{bmatrix} \right)$, then $U|V \sim \mathrm{N}(\mu_U + \Sigma_{UV}\Sigma_{VV}^{-1}(V - \mu_V), \Sigma_{UU} - \Sigma_{UV}\Sigma_{VV}^{-1}\Sigma_{VU})$.

(b) Prove that there exists κ such that for all $t \in \mathbb{Z}$, $0 < \frac{dv_{t+1}}{dv_t} \le \kappa < 1$, and use this to show that as $t \to \infty$, $v_t \to v^*$, where v^* is the unique value such that if $v_t = v^*$, then $v_{t+1} = v^*$.

(c) Find the values of λ and v^* such that if $z_t | \mathcal{F}_t \sim \mathrm{N}(m_t, v^*)$, then

$$z_{t+1} | \mathcal{F}_{t+1} \sim \mathrm{N}\big((1 - \lambda) m_t + \lambda a_{t+1}, v^*\big).$$

5) Use the properties of the Kronecker product to convert the Kim et al. (2008) representation into a VAR in $[x_{1,t}' \quad x_{1,t}' \otimes x_{1,t}' \quad x_{2,t}']'$ as in the Kollmann (2013) estimation method. For “bonus points”, derive the covariance matrix of the shock.

6) Complete the following exercises from “Methods for applied macroeconomic research” (Canova):

a)

Exercise 6.3 *(Nonlinear state space model) Consider the model* $y_t = \alpha_t + v_{1t}$, $\alpha_{t+1} = \alpha_t\theta + v_{2t}$ *and suppose one is interested in* θ*, which is unobservable, as is* α_t*. (In a trend-cycle decomposition,* θ *represents, e.g., the persistence of the trend). Cast the problem in a state space format; show the state vector and display the matrices of the model.*

b)

Exercise 6.4 *Consider a vector MA process* $y_t = e_t + e_{t-1}$ *where* $e_t \sim \mathbb{N}(0, I)$*. Show that the optimal one-step ahead predictor for* y_{t+1} *is* $y_{t+1|t} = \frac{t+1}{t+2}[y_t - y_{t|t-1}]$*. Conclude that as* $T \to \infty$*, the optimal one-step ahead predictor is just last period's forecast error. (Hint: Cast the process into a state space format and apply the Kalman filter).*

c)

Exercise 6.16 *Consider an AR(2) process* $y_t = A_0 + A_1 y_{t-1} + A_2 y_{t-2} + e_t$ *where* $e_t \sim iid\ \mathbb{N}(0, \sigma_e^2)$*. Show that the exact log likelihood function is* $\mathcal{L}(\phi) \propto -T\log(\sigma_e) + 0.5log((1+A_2)^2[(1-A_2)^2 - A_1^2]) - \frac{1+A_2}{2\sigma^2}[(1-A_2)(y_1-\bar{y})^2 - 2A_1(y_1-\bar{y})(y_2-\bar{y}) + (1-A_2)(y_2-\bar{y})^2] - \sum_{t=3}^{T} \frac{(y_t - A_0 - A_1 y_{t-1} + A_2 y_{t-2})^2}{2\sigma^2}$ *where* $\bar{y} = \frac{A_0}{1-A_1-A_2}$*. Which terms disappear if a conditional likelihood approach is used? Show that* $\sigma_{ML}^2 = \frac{1}{T-2}\sum_{t=3}^{T}(y_t - A_{0,ML} - A_{1,ML}y_{t-1} - A_{2,ML}y_{t-2})^2$*.*