Endogenous growth theory

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Outline of today's talk

- First generation endogenous growth models.
- Semi-endogenous growth models.
- Second generation endogenous growth models.

- Deeper micro-foundations for endogenous growth.
    - Quality ladders.
    - Preference for variety.

Reading for today

- A growth textbook. E.g.:
  - "Economic Growth": Barro and Sala-i-Martin
  - "The Economics of Growth": Aghion and Howitt
  - "Introduction to Modern Economic Growth": Acemoglu
- Charles Jones: "Growth with or without scale effects"
  - http://pubs.aeaweb.org/doi/pdfplus/10.1257/aer.89.2.139

Motivation

- In the models you have seen up to now, all growth was driven by exogenous movements in total factor productivity.
  - But what is total factor productivity? And why should it grow?
  - These are the questions answered by endogenous growth theory.
- While the welfare consequences of business cycles are generally small, the welfare consequences of even tiny changes in growth rates can be huge.
  - So understanding what we can do to encourage long-run growth is crucial for policy.
- Example:
  - Suppose $C_t = e^{gt+\sigma\epsilon_t-\sigma^2/2}$ where $\epsilon_t \sim \mathrm{NIID}(0,1)$, so $\mathbb{E}_{t-1}C_t = e^{gt}$.
  - And suppose household utility is given by $U_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log C_t$.
  - Then $U_0 = \sum_{t=0}^{\infty} \beta^t (gt - \sigma^2/2) = \frac{\beta g}{(1-\beta)^2} - \frac{\sigma^2}{2}\frac{1}{1-\beta}$.
  - $\beta \approx 0.99$ means $U_0 \approx 9900g - 50\sigma^2$. g is much more important!

The AK model (1/2)

- Suppose that there are not in fact decreasing returns to capital, holding fixed labour. In particular, set $Y = AKL^{1-\alpha}$. (Standard AK model has $\alpha = 1$.)
  - You might like to think of K as "human capital", or the stock of ideas/knowledge.
  - Whereas my factory cannot use your machines, it can use your ideas.
  - Knowledge is non-rival.
- Suppose labour is supplied inelastically, with each household supplying one unit, and suppose the number of households is given by $N(t) = N_0 e^{nt}$.
- Households maximise: $U = \int_0^\infty N(t) e^{-\varrho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt$, where $c(t) = \frac{C(t)}{N(t)}$ is consumption per head.
- Then:

$$U = \int_0^\infty e^{-\rho t} N_0^\sigma \frac{C(t)^{1-\sigma}}{1-\sigma} dt$$

  - where $\rho \coloneqq \varrho - \sigma n$.
- As ever, capital evolves according to: $\dot{K}(t) = I(t) - \delta K(t)$, where $Y(t) = C(t) + I(t)$.

The AK model (2/2)

- We form the current value Hamiltonian:

$$\mathcal{H}_c(K,C,\lambda) = N_0^{\sigma}\frac{C^{1-\sigma}}{1-\sigma} + \lambda\left[AKN_0^{1-\alpha}e^{(1-\alpha)nt} - C - \delta K\right].$$

- FOCs:

$$\rho\lambda - \dot{\lambda} = \mathcal{H}_{c,1}(K,C,\lambda) = \lambda A N_0^{1-\sigma} e^{(1-\alpha)nt} - \lambda\delta$$
$$0 = \mathcal{H}_{c,2}(K,C,\lambda) = N_0^{\sigma} C^{-\sigma} - \lambda$$

- So:

$$\frac{\dot{C}}{C} = \frac{AN_0^{1-\sigma}e^{(1-\alpha)nt} - \delta - \varrho}{\sigma} + n$$

- Suppose $n = 0$ or $\alpha = 1$, then $\frac{\dot{C}}{C} = \frac{AN_0^{1-\alpha} - \delta - \varrho}{\sigma} + n$, so we have exponential growth, even without growth in A, providing A is large enough.
- However, if $n > 0$ and $\alpha < 1$, then growth rates are increasing over-time, so we have super-exponential (explosive) growth.
  - $n = 0$ or $\alpha = 1$ is a “knife-edge” assumption for endogenous growth.
  - Note also that changes in the level of population (N_0) imply counter-factual changes in the rate of consumption growth.
  - This is a “strong scale effect” in Jones’s terminology.

The first generation endogenous growth model (Romer (1986), Lucas (1988), Grossman Helpman (1991), Aghion and Howitt (1992)) (1/2)

- Clearly, K in the AK model is not physical capital.
  - We are better off preserving the letter K for physical capital then.
- Is it a good model of the knowledge stock?
  - It is odd to think of knowledge as depreciating at any significant rate. With a few notable exceptions, we have access now to most of the material that has ever been published.
  - It is stranger still to think of knowledge as being produced from physical goods.
- Instead, we might directly model a productivity production function of the form $\dot{A} = \kappa A L_A$, where L_A is the labour devoted to R&D.
- Abstracting from physical capital, we suppose a production function of the form $Y = A^{\zeta} L_Y$, where $L_A + L_Y = N = N_0 e^{nt}$.
  - Define $s \coloneqq \frac{L_A}{N}$ as the fraction employed in R&D.
  - Immediately we have that $\dot{A}/A = \kappa s N_0 e^{nt}$, so there can only be exponential growth if it happens that the optimal s satisfies $\dot{s}/s = -n$, so the number engaged in R&D aren't growing over time.
  - It is also clear from this that policies designed to promote R&D have a large pay-off in this model, as an increase in s increases the growth rate.
  - We again have strong scale effects too, with N_0 increasing productivity growth rates.
- We keep household utility as before, though now $Y = C$.

The first generation endogenous growth model (2/2)

- We form the current value Hamiltonian:

$$\mathcal{H}_c(A,s,\lambda)=N_0^\sigma\frac{\left(A^\zeta(1-s)N_0e^{nt}\right)^{1-\sigma}}{1-\sigma}+\lambda\kappa sAN_0e^{nt}.$$

- FOCs:

$$\rho\lambda-\dot{\lambda}=\mathcal{H}_{c,1}(A,s,\lambda)=(1-\sigma)\zeta N_0^\sigma\frac{\left(A^\zeta(1-s)N_0e^{nt}\right)^{1-\sigma}}{1-\sigma}\frac{1}{A}+\lambda\kappa sN_0e^{nt}$$

$$0=\mathcal{H}_{c,2}(A,s,\lambda)=-(1-\sigma)N_0^\sigma\frac{\left(A^\zeta(1-s)N_0e^{nt}\right)^{1-\sigma}}{1-\sigma}\frac{1}{s}+\lambda\kappa AN_0e^{nt}$$

- Hence:

$$A\rho\lambda-A\dot{\lambda}=A(\lambda\kappa sN_0e^{nt})+\zeta s(\lambda\kappa AN_0e^{nt})=(1+\zeta)\lambda\kappa sAN_0e^{nt}.$$

- I.e.: $\dot{\lambda}/\lambda=\varrho-\sigma n-(1+\zeta)\kappa sN_0e^{nt}$
- Then from taking growth rates in the second FOC:

$$(1-\sigma)\left[\zeta\frac{\dot{A}}{A}-\frac{\dot{s}}{1-s}+n\right]-\frac{\dot{s}}{s}=\varrho-\sigma n-(1+\zeta)\kappa sN_0e^{nt}+\frac{\dot{A}}{A}+n$$

- Substituting $\dot{s}/s=-n$ makes clear this is not consistent with exponential growth in A unless $n=0$ or $\zeta=0$.

The semi-endogenous growth model of Jones (1995b)

- The key assumption driving growth in first generation endogenous growth models was the linear technology for the production of new ideas.
- But plausibly, R&D is getting harder over time as all of the obvious ideas have already been thought up.
- This suggests a knowledge production function of the form: $\dot{A} = \kappa A^{\phi} L_A$, where $\phi < 1$.
- Then: $\dot{A}/A = \kappa s A^{\phi-1} N_0 e^{nt}$, so if $n = 0$ and s is constant, then growth rates are declining over time, and growth is sub-exponential.
  - $\phi = 1$ was another implicit knife-edge assumption in the first generation models.
  - To see the problem with $\phi = 1$ another way, note that along the balanced growth path (bgp), we must have $n = (1-\phi) g_A$, i.e. $g_A \coloneqq \frac{n}{1-\phi}$.
- Semi-endogenous growth models have very different policy implications, since s no longer appears in the growth rate.
  - Thus policy cannot do much to influence long-run growth (beyond encouraging people to have more children).
  - Since the growth rate of productivity depends on the growth rate of population, we say the model has "weak scale effects".

Second generation endogenous growth models (Young (1998), Peretto (1998), Aghion and Howitt (1998), Dinopoulos and Thompson (1998), Li (2000, 2002))

- Suppose the final consumption good is produced using $C = \mathcal{J}^{1+\nu}\left[\frac{1}{\mathcal{J}}\int_0^{\mathcal{J}} Y_i^{\frac{1}{1+\mu}}\,di\right]^{1+\mu}$.
  - ν controls the returns to variety. $\nu = 0$ and $\nu = \mu$ are common choices.
- Suppose that product i is produced using the linear technology $Y_i = A_i L_{Y,i}$, where $\dot{A}_i = \kappa A_i^{\phi_A}\mathcal{J}^{\psi_A}L_{A,i}$.
- And suppose that $\mathcal{J}$ grows over time according to $\dot{\mathcal{J}} = \gamma\left(\sup_{i\in[0,\mathcal{J}]} A_i\right)^{\phi_{\mathcal{J}}-1}\mathcal{J}^{\psi_{\mathcal{J}}}L_{\mathcal{J}}$.
- We assume resources are equally allocated across varieties, and that initial conditions are identical, so we drop i subscripts in the following.
  - Let $L_A = \frac{1}{\mathcal{J}}s_A(1-s_{\mathcal{J}})N$, $L_Y = \frac{1}{\mathcal{J}}(1-s_A)(1-s_{\mathcal{J}})N$, $L_{\mathcal{J}} = s_{\mathcal{J}}N$, where the share parameters are constant along the bgp.
- Hence, $C = \mathcal{J}^{1+\nu}AL_Y = \mathcal{J}^{\nu}A(1-s_A)(1-s_{\mathcal{J}})N$, so $g_C = \nu g_{\mathcal{J}} + g_A + g_N$.
- Also, on the bgp (if it exists): $0 = (\phi_A - 1)g_A + (\psi_A - 1)g_{\mathcal{J}} + n$ and $0 = (\psi_{\mathcal{J}} - 1)g_{\mathcal{J}} + (\phi_{\mathcal{J}} - 1)g_A + n$, from the laws of motion for A and $\mathcal{J}$.
  - I.e. $\begin{bmatrix}1-\phi_A & 1-\psi_A\\ 1-\phi_{\mathcal{J}} & 1-\psi_{\mathcal{J}}\end{bmatrix}\begin{bmatrix}g_A\\ g_{\mathcal{J}}\end{bmatrix} = \begin{bmatrix}n\\ n\end{bmatrix}$.
  - If $\phi_A \neq \phi_{\mathcal{J}}$ or $\psi_A \neq \psi_{\mathcal{J}}$, then we can (probably) invert the matrix on the LHS, to find $g_A, g_{\mathcal{J}} \propto n$ (i.e. this is semi-endogenous growth).
  - If $\phi_A = \phi_{\mathcal{J}}(=\phi)$ and $\psi_A = \psi_{\mathcal{J}}(=\psi)$ then $\frac{\dot{A}/A}{\kappa s_A(1-s_{\mathcal{J}})} = \frac{\dot{\mathcal{J}}/\mathcal{J}}{\gamma s_{\mathcal{J}}} = A^{\phi-1}\mathcal{J}^{\psi-1}N_0e^{nt} = A_0^{\phi-1}\mathcal{J}_0^{\psi-1}N_0$, so R&D shares again matter, and exponential productivity growth persists whether or not population growth is zero (i.e. this is endogenous growth).

Microfoundations of endogenous growth

- In order to understand whether the knife edge assumptions behind endogenous growth are plausible, it is important to understand the mechanisms behind growth a bit more carefully.

- There are three broad classes:

  - Schumpeterian/creative destruction/quality-ladder models, used by Aghion and Howitt.

  - Variety expansion models, used by Romer.

  - Incremental improvement models, used by Peretto (and me!).
    - The second generation model we presented previously was of this class, with a production function for technology in each industry.

- Various combinations of these classes are possible.

Quality ladder models (1/7)

- Let $C = \left[\int_0^1 Y_i^{\frac{1}{1+\mu}}\, di\right]^{1+\mu}$.
- Then from the aggregators' FOC, $P_i = C^{\frac{\mu}{1+\mu}} Y_i^{-\frac{\mu}{1+\mu}}$ where P_i is the price of good i.
- Hence, $Y_i = P_i^{-\frac{1+\mu}{\mu}} C$.
- Suppose that there are J_i firms in industry i, and that firm j in industry i has the production technology $Y_{i,j} = A_{i,j} L_{Y,i,j}$.
- We suppose that there is free entry of firms to the industry, with zero entry cost.
  - Again, zero entry cost is a very special assumption.
- Suppose firms in each industry compete in price (Bertrand).
- Let $A_i \coloneqq \max_{j \in \{1,\dots,J_i\}} A_{i,j}$, $j_i \in \arg\max_{j \in \{1,\dots,J_i\}} A_{i,j}$ and $A_i^{\diamond} \coloneqq \max\{A_{i,j} \mid j \in \{1,\dots,J_i\}, j \neq j_i\}$.
- As standard, if $J_i = 1$, $P_i = (1+\mu)\frac{W}{A_i}$, where W is the wage.
- If $J_i > 1$, $P_i = \min\left\{(1+\mu)\frac{W}{A_i}, \frac{W}{A_i^{\diamond}}\right\}$ and only the firm j_i for which $A_{i,j_i} = A_i$ will produce anything.

Quality ladder models (2/7)

- Suppose that if firm j in industry i devotes $L_{R,i,j}$ units of labour to research during the interval $[t, t + dt]$, the probability that they come up with a productivity improvement in that period is $\kappa L_{R,i,j}^{\phi} dt$.
  - We will simplify by taking $\phi = 1$ in the below, but remember that without $\phi = 1$ things can be quite different.
- Suppose further that when firm j comes up with such an improvement, its new productivity is $1 + \gamma$ times the old A_i. Hence $A_i^{\diamond} = \frac{A_i}{1+\gamma}$.
- Let $\hat{\gamma} \coloneqq \min\{\gamma, \mu\}$, then $P_i = (1 + \hat{\gamma})\frac{W}{A_i}$, so $1 = \int_0^1 P_i^{-\frac{1}{\mu}} di = (1 + \hat{\gamma})^{-\frac{1}{\mu}} W^{-\frac{1}{\mu}} \int_0^1 A_i^{\frac{1}{\mu}} di$.
  - I.e. $W = \frac{1}{1+\hat{\gamma}}\left[\int_0^1 A_i^{\frac{1}{\mu}} di\right]^{\mu}$.
- And the profit flow to the incumbent in industry i with productivity A is:

$$\pi(t|A) \coloneqq \hat{\gamma}(1 + \hat{\gamma})^{-\frac{1+\mu}{\mu}} \left(\frac{W(t)}{A}\right)^{-\frac{1}{\mu}} C(t)$$

Quality ladder models (3/7)

- When they are displaced, by the free entry condition, their present discounted value must be 0, so the total value of being the incumbent in industry i at t, with productivity A is:

$$
\begin{aligned}
&V_i(t|A) \\
&= \pi(t|A)\,dt - W(t)L_{R,i,j_i}(t)\,dt \\
&+ (1 - r(t)\,dt)\big[\big(1 - \kappa\mathcal{L}_i(t)\,dt - \kappa L_{R,i,j_i}(t)\,dt\big)V_i(t + dt|A) + (\kappa\mathcal{L}_i(t)\,dt)0
\end{aligned}
$$

Quality ladder models (4/7)

- Let $O_{i,j}(t) \equiv 0$ be the value of firm j in industry i when $j \neq j_i$. Then:

$$O_{i,j}(t)$$
$$= -W(t)L_{R,i,j}(t)\,dt + (1 - r(t)\,dt)\left[\left(1 - \kappa L_{R,i,j}(t)\,dt\right)O_{i,j}(t + dt) + \left(\kappa L_{R,i,j}(t)\,dt\right)V_i(t + dt|A_i(t)(1 + \gamma))\right]$$
$$+ \omega_{i,j}(t)L_{R,i,j}(t)\,dt$$

- I.e. $W(t) = \kappa V_i(t|A_i(t)(1 + \gamma)) + \omega_{i,j}(t)$.
- Hence, from the FOC for $L_{R,i,j_i}(t)$, $\omega_{i,j_i}(t) = \kappa V_i(t|A_i(t)) + \omega_{i,j}(t) > 0$, so $L_{R,i,j_i}(t) = 0$.
- This implies that:

$$V_i(t|A) = \pi(t|A)\,dt \ + (1 - r(t)\,dt)(1 - \kappa \mathcal{L}_i(t)\,dt)V_i(t + dt|A)$$

- Hence:

$$\kappa \mathcal{L}_i(t)V_i(t|A) - \pi(t|A) = \dot{V}_i(t|A) - r(t)V_i(t|A)$$

- I.e., if $\omega_{i,j}(t) = 0$: $\mathcal{L}_i(t)W(t) - \pi(t|A_i(t)(1 + \gamma)) = \frac{1}{\kappa}\left(\dot{W}(t) - r(t)W(t)\right)$
- So:

$$\mathcal{L}_i(t) = \frac{1}{\kappa}\left(\frac{\dot{W}(t)}{W(t)} - r(t)\right) + \hat{\gamma}(1 + \hat{\gamma})^{-\frac{1+\mu}{\mu}}W(t)^{-\frac{1+\mu}{\mu}}\left(A_i(t)(1 + \gamma)\right)^{\frac{1}{\mu}}C(t)$$

- Note, $\mathcal{L}_i(t)$ is increasing in $A_i(t)$, so differences in initial conditions get amplified over time, and there is no convergence across industries.

Quality ladder models (5/7)

- To close the model, we specify households as maximising:

$$U = \int_0^\infty e^{-\rho t} \left[\log C(t) - \frac{1}{1+\nu} L(t)^{1+\nu} \right] dt$$

- Subject to the budget constraint:

$$WL + rB + \Pi = C + \dot{B}$$

- Current value Hamiltonian:

$$\mathcal{H}_c(B, [C, L], \lambda) = \log C - \frac{1}{1+\nu} L(t)^{1+\nu} + \lambda[WL + rB + \Pi - C]$$

- FOCs:

$$\rho\lambda - \dot{\lambda} = \mathcal{H}_{c,1}(B, [C, L], \lambda) = \lambda r$$

$$0 = \mathcal{H}_{c,2(1)}(B, [C, L], \lambda) = \frac{1}{C} - \lambda$$

$$0 = \mathcal{H}_{c,2(2)}(B, [C, L], \lambda) = -L^\nu + \lambda W$$

- So $\frac{\dot{\lambda}}{\lambda} = -\frac{\dot{C}}{C} = \rho - r$, and $\frac{W}{C} = L^\nu$.

Quality ladder models (6/7)

- Suppose that $A_i(0) = 1$ for all $i \in [0,1]$, then for any $i \in [0,1]$:

$$W(t) = \frac{1}{1+\hat{\gamma}} \left[\sum_{k=0}^{\infty} (1+\gamma)^{\frac{k}{\mu}} \Pr\left(A_i(t) = (1+\gamma)^k\right) \right]^{\mu}$$

- Although each individual industry has stochastic output, aggregate output will not be stochastic.
- In the limit as $\mu \to \infty$, this becomes:

$$\log W(t) = -\log(1+\gamma) + \sum_{k=1}^{\infty} k \log(1+\gamma) \Pr\left(A_i(t) = (1+\gamma)^k\right)$$

- Now, a nice property of Poisson processes is that the number of times their event happens in a fixed interval is Poisson distributed with parameter given by the integral of the rate over that time, i.e. :

$$\Pr\left(A_i(t) = (1+\gamma)^k\right) = \frac{1}{k!} e^{-\kappa \int_0^t \mathcal{L}_i(\tau)\, d\tau} \left(\kappa \int_0^t \mathcal{L}_i(\tau)\, d\tau \right)^k$$

Quality ladder models (7/7)

- We solve the tractable case in which $\mu = \infty$, since in this case $\mathcal{L}_i(t)$ is not a function of i (so, in particular, it is not a function of $A_i(t)$).
  - In fact: $\mathcal{L}(t) \coloneqq \mathcal{L}_i(t) = \frac{1}{\kappa}\left(\frac{\dot{W}(t)}{W(t)} - r(t)\right) + \frac{\gamma}{1+\gamma}\frac{C(t)}{W(t)}$
  - So: $\log W(t) = \left(\kappa \int_0^t \mathcal{L}(\tau)\,d\tau\right) e^{-\kappa \int_0^t \mathcal{L}(\tau)\,d\tau}\left[\sum_{k=1}^{\infty} \frac{\left(\kappa \int_0^t \mathcal{L}(\tau)\,d\tau\right)^{k-1}}{(k-1)!} - 1\right]\log(1+\gamma)$

- Now, on the bgp, $r - \rho = \frac{\dot{C}}{C} = \frac{\dot{W}}{W} = g$ and $L = \bar{L}$ ($g, \bar{L}$ are constants), thus providing these variables converge to the bgp quickly enough, $\frac{\kappa}{t}\int_0^t \mathcal{L}_i(\tau)\,d\tau \to \kappa \frac{\gamma}{1+\gamma}\bar{L}^{-\nu} - \rho$, so:

$$\log W(t) - t\left(\kappa \frac{\gamma}{1+\gamma}\bar{L}^{-\nu} - \rho\right)\left[e^{-t\left(\kappa\frac{\gamma}{1+\gamma}\bar{L}^{-\nu} - \rho\right)} \sum_{k=0}^{\infty} \frac{\left(t\left(\kappa \frac{\gamma}{1+\gamma}\bar{L}^{-\nu} - \rho\right)\right)^k}{k!} - e^{-t\left(\kappa\frac{\gamma}{1+\gamma}\bar{L}^{-\nu} - \rho\right)}\right]\log(1+\gamma) \to 0 \text{ as } t \to \infty$$

$$\Rightarrow \log W(t) - t\left(\kappa \frac{\gamma}{1+\gamma}\bar{L}^{-\nu} - \rho\right)\log(1+\gamma) \to 0 \text{ as } t \to \infty$$

- Hence $g = \left(\kappa \frac{\gamma}{1+\gamma}\bar{L}^{-\nu} - \rho\right)\log(1+\gamma)$.
  - Exponential growth! Note cross sectional variance of log productivity is increasing over time. Plausible?

Variety expansion models (1/4)

- We have already seen the basic idea behind variety expansion models.
  - Dixit-Stiglitz aggregators incorporate a preference for variety, so the introduction of new products raises productivity.
- The literature using variety expansion for growth often places the Dixit-Stiglitz aggregator on the production side.
  - For a bit of "variety", we present a version with investment specific technological change.
- In particular, suppose the final good is produced in a perfectly competitive industry using the technology $Y = K^{\alpha} L^{1-\alpha}$.
  - Let W be the wage and r_K the rental rate of capital. Then $W = (1-\alpha)\frac{Y}{L}$ and $r_K = \alpha\frac{Y}{K}$.
- Also suppose household capital K evolves according to $\dot{K} = I - \delta K$, where the investment good is produced from intermediate goods $M_i \in [0, \mathcal{J}]$ using the technology:

$$I = \left[\int_0^{\mathcal{J}} M_i^{\frac{1}{1+\mu}}\, di\right]^{1+\mu}.$$

- Suppose that inventing a new intermediate good requires a fixed cost of $Y_F \mathcal{J}^{\theta}$ units of the final good, and that once invented, the inventor is the only person who can produce that intermediate good, one for one from the final good.
- Market clearing requires $Y = C + \dot{\mathcal{J}} Y_F \mathcal{J}^{\theta} + \int_0^{\mathcal{J}} M_i\, di$.

Variety expansion models (2/4)

- As ever, for all $i \in [0, \mathcal{J}]$: $M_i = \left(\frac{P_i}{P}\right)^{-\frac{1+\mu}{\mu}} I$.
  - P is the price of the aggregate investment good in units of the consumption good.
- Since intermediate produces have marginal costs of 1: $P_i = 1 + \mu$.
  - Hence: $P = \left[\int_0^{\mathcal{J}} P_i^{-\frac{1}{\mu}} di\right]^{-\mu} = (1+\mu)\mathcal{J}^{-\mu}$, and $\frac{\dot{P}}{P} = -\mu \frac{\dot{\mathcal{J}}}{\mathcal{J}}$.
  - The price of the investment good is unambiguously decreasing in $\mathcal{J}$.
- Then firm profits at t are:

$$\pi(t) = \mu(1+\mu)^{-\frac{1+\mu}{\mu}} P(t)^{\frac{1+\mu}{\mu}} I(t) = \mu \mathcal{J}(t)^{-(1+\mu)} I(t)$$

- Free entry of inventors at t implies:

$$Y_F \mathcal{J}(t)^{\theta} = \pi(t)\, dt + (1 - r(t)\, dt) Y_F \mathcal{J}(t + dt)^{\theta}$$

- Hence:

$$r Y_F \mathcal{J}^{\theta} = \pi + \theta Y_F \mathcal{J}^{\theta - 1} \dot{\mathcal{J}} = \mu \mathcal{J}^{-(1+\mu)} I + \theta Y_F \mathcal{J}^{\theta-1} \dot{\mathcal{J}}$$

- I.e. $r Y_F - \theta Y_F \frac{\dot{\mathcal{J}}}{\mathcal{J}} = \mu \mathcal{J}^{-(1+\mu+\theta)} I$.

Variety expansion models (3/4)

- To close the model, we specify households as maximising:

$$U = \int_0^\infty e^{-\rho t}\left[\log C(t) - \frac{1}{1+\nu}L(t)^{1+\nu}\right]dt$$

  - Subject to the budget constraint: $WL + r_K K + rB + \Pi = C + PI + \dot{B}$.
  - And the capital accumulation constraint: $\dot{K} = I - \delta K$.

- Current value Hamiltonian:

$$\mathcal{H}_c([B,K],[C,L,I],\lambda) = \log C - \frac{1}{1+\nu}L(t)^{1+\nu} + \lambda_B[WL + r_K K + rB + \Pi - C - PI] + \lambda_K[I - \delta K]$$

- FOCs:

$$\rho\lambda_B - \dot{\lambda}_B = \mathcal{H}_{c,1(1)}([B,K],[C,L,I],\lambda) = \lambda_B r$$

$$\rho\lambda_K - \dot{\lambda}_K = \mathcal{H}_{c,1(2)}([B,K],[C,L,I],\lambda) = \lambda_B r_K - \delta\lambda_K$$

$$0 = \mathcal{H}_{c,2(1)}([B,K],[C,L,I],\lambda) = \frac{1}{C} - \lambda_B$$

$$0 = \mathcal{H}_{c,2(2)}([B,K],[C,L,I],\lambda) = -L^\nu + \lambda_B W$$

$$0 = \mathcal{H}_{c,2(3)}([B,K],[C,L,I],\lambda) = -P\lambda_B + \lambda_K$$

- So $\frac{\dot{\lambda}_B}{\lambda_B} = -\frac{\dot{C}}{C} = \rho - r$, and $\frac{W}{C} = L^\nu$.

- Also: $\rho P\lambda_B - P\dot{\lambda}_B - \dot{P}\lambda_B = \lambda_B r_K - \delta P\lambda_B$, so $r = \frac{r_K}{P} - \delta + \frac{\dot{P}}{P} = \frac{r_K}{P} - \delta - \mu\frac{\dot{\mathcal{J}}}{\mathcal{J}}$.

Variety expansion models (4/4)

- Now, on the bgp, $r = \bar{r}$, $L = \bar{L}$ and $\frac{\dot{\mathcal{J}}}{\mathcal{J}} = g_{\mathcal{J}}$.
  - $\bar{r} = \frac{r_K}{P} - \delta - \mu g_{\mathcal{J}}$, thus $\frac{\dot{r}_K}{r_K} = \frac{\dot{P}}{P} = -\mu g_{\mathcal{J}}$.
  - But $r_K = \alpha K^{\alpha-1}\bar{L}^{1-\alpha}$, hence $-\mu g_{\mathcal{J}} = \frac{\dot{r}_K}{r_K} = (\alpha - 1)\frac{\dot{K}}{K}$.
  - I.e. $\frac{\dot{K}}{K} = \frac{\mu g_{\mathcal{J}}}{1-\alpha}$, so $\frac{\dot{Y}}{Y} = \frac{\alpha}{1-\alpha}\mu g_{\mathcal{J}}$.
  - $\frac{W}{C} = \bar{L}^{\nu}$ implies $\frac{\dot{W}}{W} = \frac{\dot{C}}{C} = \bar{r} - \rho$, so as $W = (1-\alpha)K^{\alpha}\bar{L}^{-\alpha}$, $\frac{\dot{W}}{W} = \frac{\alpha}{1-\alpha}\mu g_{\mathcal{J}}$, we have that $\frac{\dot{Y}}{Y} = \frac{\dot{W}}{W} = \frac{\dot{C}}{C} = \bar{r} - \rho = \frac{\alpha}{1-\alpha}\mu g_{\mathcal{J}}$, so $\bar{r} = \rho + \frac{\alpha}{1-\alpha}\mu g_{\mathcal{J}}$.
  - Finally, since $\bar{r}Y_F - \theta Y_F g_{\mathcal{J}} = \mu \mathcal{J}^{-(1+\mu+\theta)} I$, we must have $0 = -(1+\mu+\theta)g_{\mathcal{J}} + \frac{\dot{K}}{K}$, i.e. $1 + \mu + \theta = \frac{\mu}{1-\alpha}$, so growth requires a further knife edge assumption on e.g. θ.
    - This is related to the result of Huffman (2007).
    - Including population growth would also do the trick, but would turn the model into a semi-endogenous growth one.

An immodest slide

- In my own work, I build an endogenous growth model in which there is true competition in each industry, with multiple firms producing each product, at each point in time.
- There is both free entry into an industry, and free entry of new industries, while growth comes from incremental productivity improvements performed by individual firms.
- These two margins of entry prove crucial for generating robust endogenous growth, and they allow me both to match the absence of a unit root in GDP, and to generate exponential growth even with asymmetric spill-overs from product to process innovation.